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On the Iterative Solution of the Matrix Equation $AX^2 - I = 0$

By Pentti Laasonen

Introduction. The general eigenvalue problem

$$(\lambda A - B)x = 0$$

with square matrices A and B often appears in numerical analysis, and it is frequent that A and B are symmetric and A positive definite, so that the problem is equivalent to the following one:

$$(\lambda I - C)y = 0,$$

where

$$C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}},$$

$$y = A^{\frac{1}{2}}x.$$

It is clear that C is symmetric. However, in order to avoid the solution of two special eigenvalue problems, it is desirable to have a way of forming the transforming matrix $A^{-\frac{1}{2}}$ in terms of the known matrix A without a previous determination of the eigenvalues and eigenvectors of A. Indeed, this is possible by an iterative process, based on Newton's algorithm, for any matrix A with real positive eigenvalues.

Newton's method. Two simple algorithms can be presented for the iterative solution of the scalar equation

$$ax^2 - 1 = 0$$

with a positive a; these are

$$x_{i+1} = \tfrac{3}{2}x_i - \tfrac{1}{2}ax_i^3,$$

and

$$x_{i+1} = \tfrac{1}{2}x_i + \frac{1}{2ax_i}.$$

Since the first algorithm does not always converge, the second one, which always converges for a nonvanishing initial value, will be suitably extended to a particular class of matrices. The conditions under which such extension can be accomplished are described in the following theorem.

Theorem. *Let A denote a real square matrix with real, positive eigenvalues. Then the algorithm*

$$X^{(0)} = kI \tag{1}$$

$$X^{(i+1)} = \tfrac{1}{2}X^{(i)} + \tfrac{1}{2}(AX^{(i)})^{-1}$$

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where k is a non-zero constant, generates a sequence of matrices which converges to that solution of

$$AX^2 - I = 0 \tag{2}$$

which has positive eigenvalues. Moreover, the rate of convergence is quadratic.

Preliminary remarks. Some simple properties of a class J of Jacobi matrices will be useful in establishing the results of this paper. Such matrices are of the form

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{s-1} \\ 0 & 0 & \alpha_0 & \cdots & \alpha_{s-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & & \alpha_0 \end{bmatrix} = [\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_s]. \tag{3}$$

If two such matrices (denoted by bracket expressions as above) are multiplied, then

$$\begin{aligned} &[\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_s]\cdot[\beta_0, \beta_1, \beta_2, \cdots, \beta_s] \\ &\quad = [\alpha_0\beta_0, \alpha_0\beta_1 + \alpha_1\beta_0, \alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0, \cdots, \alpha_0\beta_s + \alpha_1\beta_{s-1} + \cdots + \alpha_s\beta_0], \end{aligned} \tag{4}$$

so the class J is closed under the multiplication; moreover, multiplication of matrices of this class is commutative. Finally, the inverse of a matrix in J, if it exists, is again in J.

The following simple result will also be useful.

LEMMA. *Let*

$$\{\alpha^{(i)}\}, \quad \{\beta^{(i)}\}, \quad \{\epsilon^{(i)}\} \quad (i = 0, 1, 2, \cdots)$$

denote three sequences of real numbers. Then if they are related by the relationship

$$\alpha^{(i+1)} = \epsilon^{(i)}\alpha^{(i)} + \beta^{(i)},$$

and if the sequences $\{\beta^{(i)}\}$ and $\{\epsilon^{(i)}\}$ have limits β and 0, respectively, the sequence $\{\alpha^{(i)}\}$ converges to the limit β.

The *proof* is simple, consisting first of a finite estimate for the upper bound of $\alpha^{(i)}$, whereafter the final conclusion is evident.

Finally, this section concludes with the well known Jordan matrix theorem. This theorem states that any square matrix A can be transformed by a suitable non-singular matrix U into the Jordan normal form

$$UAU^{-1} = D = \begin{bmatrix} \Lambda_1 & & & & \\ & \Lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \Lambda_m \end{bmatrix}, \tag{5}$$

where all elements outside the diagonal submatrices Λ_μ are zero and these sub-

matrices have the form

$$\Lambda_\mu = \begin{bmatrix} \lambda_\mu & 1 & 0 & & & 0 & 0 \\ 0 & \lambda_\mu & 1 & & & 0 & 0 \\ 0 & 0 & \lambda_\mu & & & 0 & 0 \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & 0 & 0 & & & \lambda_\mu & 1 \\ 0 & 0 & 0 & & & 0 & \lambda_\mu \end{bmatrix}$$

Convergence of the algorithm. Define the matrix $Y^{(i)}$ by

$$Y^{(i)} = UX^{(i)}U^{-1}.$$

Evidently the algorithm for $Y^{(i)}$:s is defined by the formula

$$Y^{(i+1)} = \tfrac{1}{2}Y^{(i)} + \tfrac{1}{2}(DY^{(i)})^{-1}.$$

Now, suppose that one of the matrices $Y^{(i)}$, say $Y^{(r)}$, has the special Jacobi form

$$Y^{(r)} = \begin{bmatrix} \Gamma_1^{(r)} & & & & \\ & \Gamma_2^{(r)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \Gamma_m^{(r)} \end{bmatrix}, \tag{6}$$

where the square submatrices $\Gamma_\mu^{(r)}$ along the diagonal have the same orders as the corresponding submatrices Λ_μ and, moreover, belong to the class J defined by (3). It is then obvious that this particular character will be maintained by all successive matrices $Y^{(r+1)}, Y^{(r+2)}, \cdots$, and, moreover, the corresponding submatrices $\Gamma_\mu^{(i)}$ are related by

$$\Gamma_\mu^{(i+1)} = \tfrac{1}{2}\Gamma_\mu^{(i)} + \tfrac{1}{2}(\Lambda_\mu\Gamma_\mu^{(i)})^{-1} \quad (\mu = 1, 2, \cdots, m;\ i = r, r+1, \cdots). \tag{7}$$

Now, transform this equation into the form

$$\Lambda\Gamma^{(i)}(2\Gamma^{(i+1)} - \Gamma^{(i)}) = I, \tag{8}$$

where the subscripts have been deleted. Then use the notations

$$\Gamma^{(i)} = [\alpha_0^{(i)}, \alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_s^{(i)}],$$
$$\Lambda = [\lambda, 1, 0, \cdots, 0],$$

substitute into the equation (8) and apply the multiplication rule (4) in order to obtain, for the elements $\alpha_k^{(i)}$ and $\alpha_k^{(i+1)}$, the system of equations

$$\lambda\alpha_0^{(i)}(2\alpha_0^{(i+1)} - \alpha_0^{(i)}) = 1, \tag{9}$$

$$2\lambda \sum_{\nu=0}^{k} \alpha_\nu^{(i)}\alpha_{k-\nu}^{(i+1)} + 2\sum_{\nu=0}^{k-1} \alpha_\nu^{(i)}\alpha_{k-\nu-1}^{(i+1)} - \lambda\sum_{\nu=0}^{k} \alpha_\nu^{(i)}\alpha_{k-\nu}^{(i)} - \sum_{\nu=0}^{k-1} \alpha_\nu^{(i)}\alpha_{k-\nu-1}^{(i)} = 0 \tag{10}$$

$$(k = 1, 2, \cdots, s).$$

The equation (9) gives for $\alpha_0^{(i+1)}$ the formula

$$\alpha_0^{(i+1)} = \tfrac{1}{2}\alpha_0^{(i)} + \frac{1}{2\lambda\alpha_0^{(i)}}.$$

Hence, if the eigenvalue $\alpha_0^{(r)}$ of $\Gamma^{(r)}$ is positive, then the eigenvalues $\alpha_0^{(r+1)}\,\alpha_0^{(r+2)}, \cdots$ converge to the positive value $\lambda^{-\frac{1}{2}}$. The equations (10) may be used for the successive computation of the elements $\alpha_1^{(i+1)}, \alpha_2^{(i+1)}, \cdots, \alpha_s^{(i+1)}$. Indeed, if one solves the equation (10) for the element $\alpha_k^{(i+1)}$, one obtains the expression

$$\alpha_k^{(i+1)} = \left(1 - \frac{\alpha_0^{(i+1)}}{\alpha_0^{(i)}}\right)\alpha_k^{(i)} + \frac{1}{\alpha_0^{(i)}} P(\alpha_0^{(i)}, \alpha_1^{(i)}, \cdots, \alpha_{k-1}^{(i)}; \alpha_1^{(i+1)}, \cdots, \alpha_{k-1}^{(i+1)}),$$

where P is a second order polynomial of its arguments.

An appeal to the lemma mentioned above justifies the conclusion that the elements $\alpha_k^{(i)}$ tend to a definite limit, as i tends to infinity. It is easy to find that the limit values are

$$\lim_{i\to\infty} \alpha_k^{(i)} = (-1)^k \frac{(2k)!}{2^{2k}(k!)^2}\lambda^{-\frac{1}{2}-k}.$$

This implies the conclusion that the sequence of the matrices Y_i is convergent and the limit matrix has the form

$$Y = \begin{bmatrix} \Gamma_1 & & & & \\ & \Gamma_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \Gamma_m \end{bmatrix},$$

where the diagonal submatrices Γ_μ are Jacobi matrices of the form (3). The equation

$$Y = \tfrac{1}{2}Y + \tfrac{1}{2}(DY)^{-1}$$

is satisfied by this matrix, and therefore this is the solution of

$$DY^2 = I.$$

Similarly, the matrices $X^{(i)}$ tend to a limit matrix X whose transform by U is Y:

$$Y = UXU^{-1}.$$

This matrix X is obviously the solution of

(2) $$AX^2 - I = 0.$$

The assumptions made in this proof (namely, first, that one of the matrices $Y^{(i)}$ has the particular form (6); and second, that the eigenvalues of this matrix are positive) are fulfilled by choosing X_0 to be equal to kI, since in this case also Y_0 is equal to kI. Of course this diagonal matrix fulfills the requirements.

From (7) one easily derives, by using the commutative law for matrices of the class J, the equation

$$\Gamma_\mu^{(i+1)} - \Lambda^{-\frac{1}{2}} = \tfrac{1}{2}(\Gamma_\mu^{(i)})^{-1}(\Gamma_\mu^{(i)} - \Lambda^{-\frac{1}{2}})^2.$$

This equation implies the further equation

$$Y^{(i+1)} - D^{-\frac{1}{2}} = \tfrac{1}{2}(Y^{(i)})^{-1}(Y^{(i)} - D^{-\frac{1}{2}})^2,$$

or, finally,

$$X^{(i+1)} - A^{-\frac{1}{2}} = \tfrac{1}{2}(X^{(i)})^{-1}(X^{(i)} - A^{-\frac{1}{2}})^2.$$

This relationship indicates that the rate of convergence of the approximations $X^{(0)}, X^{(1)}, X^{(2)}, \cdots$ to the solution X is quadratic.

The effect of round-off errors. For the quadratic rate of convergence or for the convergence at all of the applied Newton's method, it is essential that the matrices involved have a similar structure in the sense described above; that is, their transforms by the same matrix A have a similar Jacobi structure (6). A simple counter example suffices to show that the algorithm may diverge if applied to matrices not related in this sense. In fact, this happens even if the initial matrix X_0 is arbitrarily close to the correct solution.

Such an example is provided by the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & m^{-2} \end{pmatrix}; \quad A^{-\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}.$$

Now, if the initial matrix is taken to be

$$X^{(0)} = \begin{pmatrix} 1 & \epsilon \\ 0 & m \end{pmatrix}, \quad (\epsilon \neq 0)$$

then the algorithm leads to the successive approximations

$$X^{(i)} = \begin{pmatrix} 1 & \left(\dfrac{1-m}{2}\right)^i \epsilon \\ 0 & m \end{pmatrix}, \quad (i = 1, 2, \cdots).$$

This sequence diverges, if $m > 3$, despite the arbitrary closeness of $X^{(0)}$ to the correct solution. In general, it can be shown that in this sense the algorithm (1), if carried out indefinitely, is not stable whenever the ratio of the largest to the smallest eigenvalue of A exceeds the value 9.

This observation raises the question of convergence of the above method, if it is applied numerically with truncated values. In this case, although one starts with an admissible initial matrix $X^{(0)} = kI$, the method will in general lead to a sequence of approximations which do not possess the required character, due to round-off errors. Therefore, if the method is continued indefinitely, it will probably not converge. For this reason it may be appropriate to compare during the process two successive approximations $X^{(i)}$ and $X^{(i+1)}$ and discontinue the algorithm as soon as their difference no longer decreases (by some proper norm). Due to the quadratic rate of convergence, in most cases this finite process does not involve many steps; thus, the influence of the round-off errors is likely not very great.

This is true, in particular, if the numerical calculations have been made with some additional accuracy, which can be assumed to take care of these truncation errors.

However, it seems desirable to have a method for checking the correctness of an approximation and, if necessary, to improve its accuracy. This is, in fact, possible by the following means.

Let $\bar{X}$ be an approximation for the exact solution $X = A^{-\frac{1}{2}}$ and form the error

$$\Delta = \bar{X}A\bar{X} - I.$$

If δ,

$$\delta = A^{-\frac{1}{2}} - \bar{X}, \tag{11}$$

denotes the correction which has to be added to $\bar{X}$, then A can be expressed in terms of δ and $\bar{X}$ as follows.

$$\begin{aligned} A &= (\bar{X} + \delta)^{-2} = [(I + \bar{X}^{-1}\delta)^{-1}\bar{X}^{-1}]^2 \\ &= (\bar{X}^{-1} - \bar{X}^{-1}\delta\bar{X}^{-1} + \cdots)^2 = \bar{X}^{-2} - \bar{X}^{-2}\delta\bar{X}^{-1} - \bar{X}^{-1}\delta\bar{X}^{-2} + \cdots. \end{aligned}$$

Thus,

$$\Delta = -\bar{X}^{-1}\delta - \delta\bar{X}^{-1} + \cdots.$$

The series expansions, which are convergent provided δ is so small that the eigenvalues of $\bar{X}^{-1}\delta$ are less than unity in absolute value, are to be truncated after the linear terms in δ, the error caused thereby being of second order with respect to δ. An approximation of δ within this accuracy may thus be obtained by solving the equation

$$\bar{X}^{-1}\delta + \delta\bar{X}^{-1} = -\Delta = I - \bar{X}A\bar{X}. \tag{12}$$

Accordingly, the succeeding paragraph is devoted to a study of the solvability and practical methods for solving this equation.

The equation $CV + VC = K$.

THEOREM. *Let C and K be two $(n \times n)$-matrices, all eigenvalues of C being different from zero and of the same sign. Then the equation*

$$CV + VC = K \tag{13}$$

has one and only one solution V.

In order to prove this theorem, transform all matrices involved by a non-singular matrix U,

$$\begin{aligned} D &= UCU^{-1}, \\ L &= UKU^{-1}, \\ W &= UVU^{-1}, \end{aligned}$$

such that $D = (d_{i,j})$ attains any Jacobi form, that is, all elements below the diagonal, $d_{i,j}$ with $i < j$, vanish; the diagonal elements $d_{i,i}$ thus obtained are the eigenvalues of C and therefore of same sign, say positive. Accordingly, the equation is equivalent to the equation

$$DW + WD = L.$$

By setting corresponding elements on both sides of this equation equal to one another and using a self-explanatory notation, one obtains the following system of n^2 equations:

$$d_{i,i}w_{i,j} + \sum_{k=i+1}^{n} d_{i,k}w_{k,j} + \sum_{k=1}^{j-1} w_{i,k}d_{k,j} + w_{i,j}d_{j,j} = l_{i,j}\cdot \quad (i, j = 1, 2, \cdots, n).$$

Since the coefficient $d_{i,i} + d_{j,j}$ of $w_{i,j}$ is positive, this equation can be considered as a recurrence formula which determines $w_{i,j}$ in terms of the elements on the left lower side of the line through $w_{i,j}$ and parallel to the diagonal. Accordingly, all elements can be determined successively, beginning from the lower left hand corner.

Of course, the numerical solution of the equation (13) can be achieved either by solving, for the elements of V, the linear system which is obtained by setting corresponding elements on both sides of (13) equal to one another, or by first performing the transformations used in the above proof. However, the first method is quite laborious for an arbitrary C, since the number of simultaneous equations involved is n^2. On the other hand, the second method involves actually the diagonalization of the matrix C.

A third method, based on an iteration process, is certainly preferable. The equation (13) is then written as follows:

$$\left(C + \frac{1}{m}V\right)^2 = C^2 + \frac{1}{m}K + \frac{1}{m^2}V^2, \tag{14}$$

where m is any scaling factor. If m is large enough, then a good approximation for $C + \frac{1}{m}V$ is provided by the matrix $\left(C^2 + \frac{1}{m}K\right)^{\frac{1}{2}}$. Subtract C from this matrix in order to obtain an approximation for $\frac{1}{m}V$. This can now, if necessary, be substituted on the right hand side of (14) and the process continued.

The extraction of the square-root of the matrix on the right hand side of (14) can, of course, be achieved by Newton's method in a way similar to the one used for computing the matrix $A^{-\frac{1}{2}}$. For the equation

$$X^2 - A = 0,$$

where A is a matrix with non-negative eigenvalues, that solution X which has no negative eigenvalues is obtained as the limit matrix from the algorithm

$$X_{i+1} = \tfrac{1}{2}X_i + \tfrac{1}{2}AX_i^{-1}, \tag{15}$$

if the first matrix is $X_0 = kI$ with any positive k.

The determination of the correction δ. In applying the method outlined above to the determination of the correction (11) from the equation (12), observe that the neglected terms are of the order $[(\bar{X}^{-1}\delta)^2]$. Accordingly, the truncated equation (12) should properly be replaced by

$$\bar{X}^{-1}\delta + \delta\bar{X}^{-1} = -\Delta + 0[(\bar{X}^{-1}\delta)^2]$$

and therefore the corresponding equation (14) is actually

$$\left(\bar{X}^{-1} + \frac{1}{m}\,\delta\right)^2 = \bar{X}^{-2} - \frac{1}{m}\,\Delta + \frac{1}{m}\,0[(\bar{X}^{-1}\delta)^2] + \frac{1}{m^2}\,\delta^2.$$

Now, since the third term of the right hand side of this equation is to be neglected, one concludes that the removal of the last term does not cause any additional inaccuracy, if m is large enough. It is sufficient to choose m of the same order of magnitude as the largest eigenvalue of $\bar{X}$ squared or the reciprocal of the smallest eigenvalue of A. Compute the matrix $\left(\bar{X}^{-2} - \frac{1}{m}\,\Delta\right)^{\frac{1}{2}}$ by means of the algorithm (15); subtract $\bar{X}^{-1}$ from this result; and finally, multiply by m in order to obtain an approximation for the correction (11). Of course, this procedure may be repeated and thus set up a quadratically convergent algorithm which, moreover, is self-correcting.

Summary. It has been proved that for any real $(n \times n)$-matrix A with only positive eigenvalues the algorithm (1), with an initial matrix $X^{(0)} = kI$, converges quadratically to the matrix $A^{-\frac{1}{2}}$ with positive eigenvalues.

In a numerical case this algorithm, if continued indefinitely, may be divergent due to round-off errors, whose influence may increase in geometrical progression. This makes it necessary to stop the process as soon as the difference between two successive results no longer decreases; of course, it is also desirable to have some additional accuracy in the numerical computation to take care of the round-off errors.

Any approximation to $A^{-\frac{1}{2}}$ sufficiently accurate can be improved successively, the rate of convergence of the procedure being quadratic. Each step, however, involves either the solution of a system of n^2 linear equations or the extraction of the square-root of a matrix, which may be achieved by a quadratically convergent iteration process.

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On Gauss' Speeding Up Device in the Theory of Single Step Iteration

By Alexander M. Ostrowski

1. In this paper we consider throughout *real* numbers, vectors and matrices. In order to solve the linear system

$$(1) \qquad \sum_{\nu=1}^{n} a_{\mu\nu}x_\nu = y_\mu, \quad a_{\mu\mu} = A_\mu \quad (\mu = 1, \cdots, n)$$

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with the matrix A, $A_\mu \neq 0$ $(\mu = 1, \cdots, n)$ and non-vanishing determinant, by the *single step iteration* we form, starting from an arbitrary (row) vector ξ_0, a sequence of vectors

$$\xi_k = (x_1^{(k)}, \cdots, x_n^{(k)}) \quad (k = 1, 2, \cdots) \tag{2}$$

obtained in the following way: For any integer $k (k \geqslant 0)$ choose a value N_k of the "leading index" from the indices $1, \cdots, n$; then if

$$\rho_k = (r_1^{(k)}, \cdots, r_n^{(k)}) \quad (k = 0, 1, 2, \cdots) \tag{3}$$

is the k-th *residual vector* defined by

$$r_\mu^{(k)} = \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu^{(k)} - y_\mu \quad (\mu = 1, \cdots, n; k = 0, 1, \cdots), \tag{4}$$

we put

$$x_\mu^{(k+1)} = x_\mu^{(k)} \quad (\mu \neq N_k), \qquad x_{N_k}^{(k+1)} = x_{N_k}^{(k)} - \frac{r_{N_k}^{(k)}}{A_{N_k}}. \tag{5}$$

In studying this iteration we can and will restrict ourselves to the case where all y_μ vanish, since we can always by a convenient change of the origin make all $y_\mu = 0$, without changing the ρ_k.

2. We consider in what follows only the case where the matrix of the system (1) is *symmetric* and the quadratic form

$$K(\xi) = \sum_{\mu,\nu=1}^{n} a_{\mu\nu} x_\mu x_\nu = \xi A \xi' \tag{6}$$

defined for an arbitrary vector $\xi = (x_1, \cdots, x_n)$, is *positive definite*. In this case it is well known and immediately verified that if the vector ξ_{k+1} is obtained from the vector ξ_k by the transformation (5), we have

$$K(\xi_{k+1}) = K(\xi_k) - (r_{N_k}^{(k)})^2 / A_{N_k}. \tag{7}$$

In using (7) it was proved by Seidel, 1874 [13], that the single step iteration is always convergent if N_k is chosen at each step so that

$$(r_{N_k}^{(k)})^2 / A_{N_k} = \max_\mu (r_\mu^{(k)})^2 / A_\mu \tag{8}$$

This is Seidel's relaxation procedure.

This special rule goes back to F. R. Helmert, 1872 [7]. The relaxation rule indicated previously by Gauss [4] and Gerling [6] is different, as is the one proposed by Southwell [14], but the rule (8) is apparently the most advantageous one. Cf. the discussion in [9], p. 158–9.

On the other hand Schmeidler [12] and Reich [11] proved in using (7) that the single step procedure is convergent in the *cyclic* case when N_k runs periodically through all indices $1, \cdots, n$.

3. Gauss [4], [1], [6] and [15] proposed the following modification of the

above procedure in order to speed up the convergence. Put

$$x_\nu = z_\nu - z_0, \quad a_{0\nu} = a_{\nu 0} = -\sum_{\mu=1}^{n} a_{\mu\nu} \quad (\nu = 1, \cdots, n)$$

(9)

$$a_{00} = A_0 = -\sum_{\nu=1}^{n} a_{0\nu} = \sum_{\mu,\nu=1}^{n} a_{\mu\nu},$$

where z_0 can be arbitrarily chosen. Then the system (1) can be written in the form (assuming $y_\mu = 0$)

$$\sum_{\nu=0}^{n} a_{\mu\nu} z_\nu = 0 \quad (\mu = 0, 1, \cdots, n), \tag{10}$$

where the first equation is, of course, not independent of the last n equations but is useful for the sake of uniformity and for checking purposes.

In particular A_0 is positive since by (9) A_0 is the value of the quadratic form (6) for $x_\nu = 1$ ($\nu = 1, \cdots, n$).

4. From a solution $(z_0, z_1, \cdots, z_n)$ of the system (10) we obtain at once by (9) the solution $(x_1, \cdots, x_n)$ of the system (1). The idea of Gauss is now to apply the procedure described in (4) and (5) to the system (10). If we obtain then, starting from a vector $\zeta_0 = (z_0^{(0)}, z_1^{(0)}, \cdots, z_n^{(0)})$, a sequence of vectors

$$\zeta_k = (z_0^{(k)}, z_1^{(k)}, \cdots, z_n^{(k)}),$$

we consider at the same time the corresponding vectors

$$\xi_k = (z_1^{(k)} - z_0^{(k)}, \cdots, z_n^{(k)} - z_0^{(k)}).$$

If then in the passage from ζ_k to ζ_{k+1} the leading index $N_k \neq 0$, we have

$$\sum_{\nu=0}^{n} a_{\mu\nu} z_\nu^{(k)} = \sum_{\nu=1}^{n} a_{\mu\nu}(z_\nu^{(k)} - z_0^{(k)}) = r_\mu^{(k)} \quad (\mu = 1, \cdots, n), \tag{11}$$

$$z_{N_k}^{(k+1)} = z_{N_k}^{(k)} - \frac{\sum_{\nu=0}^{n} a_{N_k\nu} z_\nu^{(k)}}{A_{N_k}} = z_{N_k}^{(k)} - \frac{r_{N_k}^{(k)}}{A_{N_k}},$$

$$z_\mu^{(k+1)} = z_\mu^{(k)} \quad (\mu \neq N_k).$$

Since here $z_0^{(k+1)} = z_0^{(k)}$, we see that the corresponding n-dimensional vectors ξ_k, ξ_{k+1} are connected exactly by the formulae (5), so that in this case there is no essential change compared with the original method.

5. If, however, $N_k = 0$, then only $z_0^{(k)}$ is changed and therefore *all components* $x_1, \cdots, x_n$ are changed by the same amount. In this case we have obviously a new possibility and the question arises whether in this case the convergence is indeed speeded up. Of course, by 'convergence' in this case is not meant the convergence of the vectors ζ_k but that of the corresponding vectors ξ_k. This question is ap-

parently not as yet settled, as widely contradictory opinions are to be found in the literature; see [14] and [3].

Our conclusions are stated at the start of section 8 and at the end of the paper.

6. In what follows we will say that the two $(n+1)$-dimensional vectors $\zeta = (z_0, z_1, \cdots, z_n)$ and $\zeta' = (z_0', z_1', \cdots, z_n')$ are equivalent, if we have $z_\nu - z_0 = z_\nu' - z_0'$. In the class of vectors equivalent to ζ there exists a *reduced* one: $\hat{\zeta} = (0, x_1, \cdots, x_n)$, and the corresponding n-dimensional vector $\xi = (x_1, \cdots, x_n)$ is uniquely determined.

If we define the component of the residual vector corresponding to the index 0 by

$$r_0^{(k)} = \sum_{\nu=0}^{n} a_{0\nu} z_\nu^{(k)} = \sum_{\nu=1}^{n} a_{0\nu}(z_\nu^{(k)} - z_0^{(k)}) = -\sum_{\nu=1}^{n} r_\nu^{(k)}, \tag{12}$$

we see from (11) and (12) that the residual vector for the system (10) does not depend on the component z_0 but only on the corresponding vector ξ. It follows then from (9) and (6)

$$\begin{aligned}\sum_{\mu,\nu=0}^{n} a_{\mu\nu} z_\mu z_\nu &= \sum_{\mu=0}^{n} z_\mu \left(\sum_{\nu=1}^{n} a_{\mu\nu} z_\nu + a_{\mu 0} z_0 \right) \\ &= \sum_{\mu=0}^{n} z_\mu \sum_{\nu=1}^{n} a_{\mu\nu} x_\nu = \sum_{\nu=1}^{n} x_\nu \sum_{\mu=0}^{n} a_{\mu\nu} z_\mu \\ &= \sum_{\nu=1}^{n} x_\nu \left(\sum_{\mu=1}^{n} a_{\mu\nu} z_\mu + a_{0\nu} z_0 \right) \\ &= \sum_{\nu=1}^{n} x_\nu \sum_{\mu=1}^{n} a_{\mu\nu} x_\mu,\end{aligned}$$

$$\sum_{\mu,\nu=0}^{n} a_{\mu\nu} z_\mu z_\nu = \sum_{\mu,\nu=1}^{n} a_{\mu\nu} x_\mu x_\nu. \tag{13}$$

7. It is obvious that the algebraic identity corresponding to (7) remains true for the system (10), although the corresponding quadratic form is only *semi-definite*. Therefore from (13) it follows that the relation (7) is also true for $N_k = 0$ where $r_0^{(k)}$ is given by (12), and the quadratic form K is the positive definite quadratic form (6). But then it follows from (7) that in any case

$$\lim_{k\to\infty} (r_{N_k}^{(k)})^2 / A_{N_k} = 0. \tag{14}$$

8. We discuss first the cyclic one step iteration. In this case we will prove that the procedure remains convergent, but for any $n \geqslant 2$ there exist matrices for which the modified procedure is slower and others for which the modified procedure is indeed faster than the original one. This agrees with results mentioned by Forsythe and Motzkin [3], footnote 24.

9. It follows from (14) that $\lim r_{N_k}^{(k)} = 0$. Therefore from (5) and the corresponding formula for ζ_k with $N_k = 0$,

$$x_\mu^{(k+1)} - x_\mu^{(k)} \to 0 \quad (k \to \infty\,;\, \mu = 1, \cdots, n),$$

and by (4)

$$r_\mu^{(k+1)} - r_\mu^{(k)} \to 0 \quad (k \to \infty\,;\, \mu = 1, \cdots, n).$$

More generally for each constant integer γ,

$$r_\mu^{(k+\gamma)} - r_\mu^{(k)} \to 0 \quad (k \to \infty\,;\, \mu = 1, \cdots, n).$$

But for any fixed μ, among the $n + 1$ consecutive values of k there is one for which $N_k = \mu$; therefore it follows that

$$r_\mu^{(k)} \to 0 \quad (k \to \infty\,;\, \mu = 1, \cdots, n),$$

and, since the determinant in (4) does not vanish,

$$x_\mu^{(k)} \to 0 \quad (k \to \infty\,;\, \mu = 1, \cdots, n).$$

We see that the *modified procedure* is indeed *convergent*.

10. In comparing the rate of convergence of the original and the modified procedure it is better to change our notations in the following way. If we start with a vector $\xi_0 = X_0$ and apply the complete n-cycle of single steps corresponding to $N_k = 1, \cdots, n$, the obtained vector will be denoted by X_1, and the vectors obtained in repeating each time the complete n-cycle will be denoted by X_2, $X_3, \cdots, X_k = \xi_{nk}, \cdots$.

In the same way, in the modified cyclic procedure we obtain, starting from a vector $\zeta_0 = Z_0$ and applying each time the *whole* $(n + 1)$*-cycle* corresponding to $N_k = 0, 1, \cdots, n$, the sequence of vectors $Z_1, Z_2, \cdots, Z_k = \zeta_{nk}, \cdots$.

11. We decompose A in the following way:

$$A = L + D + L^*, \tag{15}$$

where D is the diagonal matrix with the elements $a_{11}, \cdots, a_{nn}$, while in L all elements to the right and on the main diagonal and in L^* all elements to the left and on the main diagonal vanish. Then the rate of convergence of the usual cyclic single step iteration depends on the maximum modulus λ_N of the roots of the equation

$$|\lambda(L + D) + L^*| = 0, \tag{16}$$

and we have, if $\lambda_N > 0$:

$$X_k = O(\lambda_N^k k^{n-2}) \quad (k \to \infty), \tag{17}$$

while the starting vector X_0 can be chosen so that $X_k \lambda_N^{-k}$ does not tend to 0 as $k \to \infty$. The proof of this is quite similar to the following proof of the corresponding results for the modified single step iteration; see (25). If $\lambda_N = 0$, then X_1 vanishes identically, and the solution is obtained at the most in n steps.

12. We will now characterize in a similar way the rate of convergence of the

modified procedure. We have for the matrix $\hat{A}$ of the system (10) the decomposition corresponding to (15):

$$\hat{A} = \begin{pmatrix} a_{00} & a_{0\nu} \\ a_{\nu 0} & A \end{pmatrix} = \hat{L} + \hat{D} + \hat{L}^* \tag{18}$$

and obtain between Z_0 and Z_1, as in the theory of the usual cyclic one step iteration, the relations

$$\begin{aligned} (\hat{L} + \hat{D})Z_1 + \hat{L}^*Z_0 &= 0, \\ Z_1 &= -(\hat{L} + \hat{D})^{-1}\hat{L}^*Z_0. \end{aligned} \tag{19}$$

13. As has been mentioned above the result of this operation is not changed if Z_0 is replaced by the corresponding reduced vector $\hat{Z}_0 = (0, x_1^{(0)}, \cdots, x_n^{(0)})$. Before we go on from Z_1, we replace therefore Z_1 again by the corresponding reduced vector $\hat{Z}_1 = (0, x_1^{(1)}, \cdots, x_n^{(1)})$. For this purpose we apply the transformation $x_\mu = Z_\mu - Z_0$ $(\mu = 1, \cdots, n)$ which is equivalent to left multiplication by the $(n + 1) \times (n + 1)$-matrix

$$N_0 = \begin{pmatrix} 0 & 0 \\ -1 & E_n \end{pmatrix}, \tag{20}$$

where the first row consists of 0's and the first column, with the exception of the 0 on the top, of -1's.

We have then finally, putting

$$Q_0 = -N_0(\hat{L} + \hat{D})^{-1}\hat{L}^*, \tag{21}$$

$$\hat{Z}_k = Q_0^k \hat{Z}_0 \quad (k = 1, 2, \cdots). \tag{22}$$

14. If Q_0 vanishes identically, we have at once $Z_1 = \cdots = Z_k = \cdots = 0$, and the solution of (1) is attained at the first step. We can and will therefore assume that $Q_0 \neq 0$. We use then the following result due to Werner Gautschi [5].

If for any matrix $C = (c_{\mu\nu})$ we define as its "norm"

$$N(C) = \sqrt{\sum_{\mu,\nu} |c_{\mu\nu}|^2},$$

then if C is a square matrix of order n for which the greatest modulus of a fundamental root is Λ, we have

$$N(C^k) = O(\Lambda^k k^{p-1}) \quad (k \to \infty), \tag{23}$$

where p is the greatest multiplicity of a fundamental root of C with the modulus Λ. As a matter of fact (23) is not the *best* possible result, cf. [10], p. 5, Satz V. But Werner Gautschi's result is completely sufficient for our purpose.

15. If we apply this to the singular matrix (21) and denote the maximal modulus of a fundamental root of Q_0 by λ_G, then p does not exceed $n - 1$ if $\lambda_G > 0$, as will follow later from (31). We have therefore

$$N(Q_0^k) = O(\lambda_G^k k^{n-2}) \quad (k \to \infty). \tag{24}$$

Further it follows from (22), in applying the Cauchy-Schwarz inequality,

$$|\hat{Z}_k| \leqslant N(Q_0^k)|Z_0|,$$

and we obtain therefore

$$\hat{Z}_k = O(\lambda_G^k k^{n-2}) \quad (k \to \infty). \tag{25}$$

16. On the other hand it is easy to show that for a conveniently chosen starting vector Z_0 the expression $Z_k\lambda_G^{-k}$ does not tend to zero. Indeed, if η is an eigenvector of Q_0 corresponding to a fundamental root λ with $|\lambda| = \lambda_G$, we have

$$\lambda\eta' = Q_0\eta'$$

and iterating

$$\lambda^k\eta' = Q_0^k\eta'.$$

But, since the first row in Q_0 consists of zeros, the vector η is a reduced one and can be taken as $\hat{Z}_0$. Then we have

$$\hat{Z}_k = \lambda^k\hat{Z}_0, \quad \hat{Z}_k\lambda_G^{-k} = \left(\frac{\lambda}{\lambda_G}\right)^k \hat{Z}_0,$$

and this does not tend to zero as $k \to \infty$. If $\lambda_G = 0$, then Z_1 vanishes identically.

17. We shall now transform the fundamental equation of Q_0, and we introduce for this purpose the matrix

$$N_\epsilon = \begin{pmatrix} \epsilon & 0 \\ -1 & E_n \end{pmatrix}, \tag{26}$$

where the first row and the first column consist respectively of 0's and -1's with the exception of the first element ϵ. N_ϵ corresponds to the transformation

$$y_0 = \epsilon Z_0, \quad y_\nu = Z_\nu - Z_0 \quad (\nu = 1, \cdots, n) \tag{27}$$

and for $\epsilon \to 0$ goes into N_0. Since the inverse of (27) is for $\epsilon \neq 0$

$$Z_0 = \frac{1}{\epsilon} y_0, \quad Z_\nu = y_\nu + \frac{1}{\epsilon} y_0 \quad (\nu = 1, \cdots, n),$$

we have

$$N_\epsilon^{-1} = \begin{pmatrix} \epsilon^{-1} & 0 \\ \epsilon^{-1} & E_n \end{pmatrix}, \tag{28}$$

where the first column consists of ϵ^{-1}, while the first row with the exception of the first element contains only 0's.

The fundamental equation of Q_0 can be written in the form

$$\lim_{\epsilon \to 0} |\lambda E + N_\epsilon(\hat{L} + \hat{D})^{-1}\hat{L}^*| = 0. \tag{29}$$

18. On the other hand we have identically, since $|N_\epsilon| = \epsilon$,

$$(|\hat{L} + \hat{D}|)(|\lambda E + N_\epsilon(\hat{L} + \hat{D})^{-1}\hat{L}^*|) = \epsilon|\lambda(\hat{L} + \hat{D})N_\epsilon^{-1} + \hat{L}^*|. \tag{30}$$

and we obtain the fundamental equation of Q_0 in taking the limit for $\epsilon \to 0$ on the right in (30).

Now we have

$$(\hat{L} + \hat{D})N_\epsilon^{-1} = \begin{pmatrix} \epsilon^{-1}a_{00} & 0 \\ \epsilon^{-1}\sum_{\mu \le \nu} a_{\nu\mu} & L + D \end{pmatrix},$$

where $a_{n0} + a_{n1} + \cdots + a_{nn} = 0$ by (9); and we have therefore identically

$$\epsilon|\lambda(\hat{L} + \hat{D}) + \hat{L}^*| = \lambda\begin{vmatrix} a_{00} & a_{0\nu} \\ \sum_\mu a_{\nu\mu} & \lambda(L + D) + L^* \end{vmatrix}.$$

λ_G is therefore the maximum modulus of the roots of the equation of degree n

$$G(\lambda) = \begin{vmatrix} a_{00} & a_{0\nu} \\ \sum_{\mu \le \nu} a_{\nu\mu} & \lambda(L + D) + L^* \end{vmatrix} = 0. \tag{31}$$

One root of this equation is zero.

19. In specializing for $n = 2$ we obtain in particular, if we put $a_{11} = a_1$, $a_{22} = a_2$, $a_{12} = a_{21} = \sigma$ and assume $\sigma \neq 0$:

$$\begin{aligned} G_2(\lambda) &= \begin{vmatrix} a_1 + a_2 + 2\sigma & -a_1 - \sigma & -a_2 - \sigma \\ -\sigma & \lambda a_1 & \sigma \\ 0 & \lambda\sigma & \lambda a_2 \end{vmatrix} \\ &= (a_1 + a_2 + 2\sigma)a_2 a_1 \lambda^2 - \sigma\lambda(a_1 + \sigma)(a_2 + \sigma), \end{aligned}$$

$$\lambda_G = \frac{|\sigma(a_1 + \sigma)(a_2 + \sigma)|}{a_1 a_2(a_1 + a_2 + 2\sigma)}, \tag{32}$$

while the equation (16) for λ_N reduces to $\lambda^2 a_1 a_2 - \lambda\sigma^2 = 0$, and gives

$$\lambda_N = \frac{\sigma^2}{a_1 a_2}. \tag{33}$$

From (32) and (33) we have

$$\frac{\lambda_G}{\lambda_N} = \frac{|a_1 + \sigma|\,|a_2 + \sigma|}{|\sigma|\,(a_1 + a_2 + 2\sigma)}. \tag{34}$$

20. If we square this, subtract 1 and multiply by the square of the denominator we obtain

$$\begin{aligned} &[(a_1 + \sigma)(a_2 + \sigma) - \sigma(a_1 + a_2) - 2\sigma^2][(a_1 + \sigma)(a_2 + \sigma) + \sigma(a_1 + a_2) + 2\sigma^2] \\ &\qquad = (a_1 a_2 - \sigma^2)(a_1 a_2 + 2(a_1 + a_2)\sigma + 3\sigma^2). \end{aligned}$$

Since the first factor is positive, we see that $\lambda_G \gtreqless \lambda_N$ according as

$$\varphi(\sigma) \equiv 3\sigma^2 + 2(a_1 + a_2)\sigma + a_1 a_2 \gtreqless 0. \tag{35}$$

Here $\sigma \neq 0$ is subject only to the condition $|\sigma| \leqslant \sqrt{a_1 a_2}$. We have obviously

$$\varphi(-\sqrt{a_1 a_2}) = 4\sqrt{a_1 a_2}\left(\sqrt{a_1 a_2} - \frac{a_1 + a_2}{2}\right).$$

Since this is $\leqslant 0$, we see that $\varphi(-\sigma)$ has two positive roots σ_1, σ_2 with $\sigma_1\sigma_2 = \dfrac{a_1 a_2}{3}$, and

$$0 < \sigma_1 < \sqrt{a_1 a_2} \leqslant \sigma_2.$$

It follows that for $n = 2$ we have $\lambda_G < \lambda_N$, if $\sigma_1 < -\sigma < \sqrt{a_1 a_2}$, and $\lambda_G > \lambda_N$, if $-\sqrt{a_1 a_2} < -\sigma < \sigma_1$. The modified procedure is in the first case faster and in the second slower than the original one.

21. To prove the corresponding result for $n > 2$ consider the matrix A of the quadratic form

$$K(\xi) = a_1 x_1^2 + 2\sigma x_1 x_2 + a_2 x_2^2 + \sum_{\mu=3}^{n} x_\mu^2. \tag{36}$$

In the corresponding determinant (31) for $G(\lambda)$ the elements in the first column are

$$a_{\mu 0} + a_{\mu 1} + \cdots + a_{\mu\mu} = -(a_{\mu,\mu+1} + \cdots + a_{\mu n})$$

and vanish therefore for $\mu \geqslant 2$. The same is true for the elements $\lambda a_{\mu\nu}$ to the left of the main diagonal with $\mu > 2$ and $\nu < \mu$, while the elements on the diagonal, $\lambda a_{\mu\mu}$ $(\mu > 2)$, become respectively λ. We obtain therefore

$$G(\lambda) = \lambda^{n-2} G_2(\lambda)$$

so that λ_G is given in this case by (32).

22. In the same way it follows from (16) that λ_N is again given by (33). We can have in this case, according to the chosen values of σ, either $\lambda_G > \lambda_N$ or $\lambda_G < \lambda_N$.

It may finally be remarked that the value of λ_G is not changed, if the $(n+1)$-st equation in (10) and the corresponding new variable z_0 are not put at the beginning but are interpolated between two indices μ, $\mu + 1$ or even put at the end. Indeed this amounts to the old process applied to a transformation of $\hat{\zeta}_0$ by a finite sequence of single step iterations, but then $\hat{\zeta}_0$ is carried over into the *general* reduced vector, and the invariance of λ_G follows then from the characterization of λ_G contained in the developments of numbers 15 and 16.

23. We consider now Seidel's relaxation procedure (8). Then in the modified procedure we obtain the speeding up for an index k for which we have

$$(r_0^{(k)})^2/A_0 > (r_\mu^{(k)})^2/A_\mu \quad (\mu \neq 0), \tag{37}$$

that is to say, by (12):

$$\Big|\sum_{\nu=1}^{n} r_\nu^{(k)}\Big|/A_0^{\frac12} > |r_\mu^{(k)}|/A_\mu^{\frac12} (\mu \neq 0). \tag{38}$$

We will now show that this inequality is impossible, if we have

$$A_0^{\frac12} \geq \sum_{\nu=1}^{n} A_\nu^{\frac12} - \min_{1 \leqslant \mu \leqslant n} A_\mu^{\frac12}. \tag{39}$$

Indeed, if we put

$$|r_\mu^{(k)}| = p_\mu A_\mu^{\frac{1}{2}}, \quad \mu = 1, \cdots, n,$$

we have from (38) and (39), since one of the r_μ and therefore one of the p_μ vanish:

$$\Big|\sum_{\nu=1}^{n} r_\nu^{(k)}\Big|/A_0^{\frac{1}{2}} \leqslant \sum_{\nu=1}^{n} p_\nu (A_\nu/A_0)^{\frac{1}{2}} \leqslant A_0^{-\frac{1}{2}}(-\min_{1\leqslant\mu\leqslant n} A_\mu^{\frac{1}{2}} + \sum_{\nu=1}^{n} A_\nu^{\frac{1}{2}}) \max p_\mu$$

$$\leqslant \max p_\mu = \max_{1\leqslant\mu\leqslant n} (|r_\mu^{(k)}| A_\mu^{-\frac{1}{2}}).$$

Therefore, in order that (37) be possible at all we must have

$$\sqrt{A_0} < \sum_{\mu=1}^{n} \sqrt{A_\mu} - \min_{1\leqslant\mu\leqslant n} \sqrt{A_\mu}. \tag{40}$$

24. In order to discuss the situation under the condition (40) we consider the $r_\mu^{(k)}$ as *stochastic variables* and discuss the probability for (37) under suitable assumptions on the distribution of the $r_\mu^{(k)}$. We put

$$\beta_\nu = \sqrt{A_\nu} > 0 \quad (\nu = 0, 1, \cdots, n) \tag{41}$$

and denote the sequence of the vectors $\rho_k (k = 0, 1, \cdots)$ by S. In the following formulae the index of ρ, that is the upper index of $r_\nu^{(k)}$, will be dropped whenever it is possible without danger of misunderstanding.

Then our problem can be described as the problem of computing the probability

$$P\left[\frac{1}{\beta_0}\left|\sum_{\nu=1}^{n} r_\nu\right| > \max_{1\leqslant\nu\leqslant n} \frac{|r_\nu|}{\beta_\nu}; \quad \rho \in S\right] \tag{42}$$

defined in the usual way as the limit of the relative density, and ascertaining whether (42) is positive. As the relations in the brackets of (42) are homogeneous in the r_ν, each of the vectors $\rho \in S$ can and is from now on assumed to be normed in such a way that we have

$$\max_{1\leqslant\nu\leqslant n} \frac{|r_\nu|}{\beta_\nu} = 1. \tag{43}$$

25. Denote the N-th section $(\rho_1, \cdots, \rho_N)$ of S by $S^{(N)}$ and put

$$\Phi^{(N)}(\sigma) = P\left[\sum_{\nu=1}^{n} r_\nu \leqslant \sigma, \quad \rho \in S^{(N)}\right]. \tag{44}$$

This is the "finite" probability in the classical sense, and we put then

$$\Phi(\sigma) = \lim_{N\to\infty} \Phi^{(N)}(\sigma), \tag{45}$$

if this limit exists.

For any k, $k = 1, \cdots, n$, we denote by $S_k^{(N)}$ the sequence of the vectors ρ_λ, $\lambda \leqslant N$, for which k is leading index corresponding to $\rho_{\lambda-1}$ and therefore $r_k^{(\lambda)} = 0$.

Further we write S_k for the complete partial sequence of S corresponding to these vectors.

We denote further by $L_\mu(N)$ the number of vectors ρ_λ in $S^{(N)}$ whose leading index is μ and put

$$\Phi_k^{(N)}(\sigma) = P\left[\sum_{\nu=1}^{n} r_\nu \leqslant \sigma; \quad \rho \in S_k^{(N)}\right]. \tag{46}$$

Denote now by $S_{k,\mu}$ the partial sequence of S containing the vectors ρ_λ such that μ is leading index for ρ_λ and k is leading index for $\rho_{\lambda-1}$, and therefore by (8) and (43) $r_k^{(\lambda)} = 0$, $|r_\mu^{(\lambda)}| = \beta_\mu$, and by $S_{k,\mu}^{(N)}$ the sequence of the vectors ρ_λ from $S_{k,\mu}$ with $\lambda \leqslant N$. Put then

$$\Phi_{k,\mu}^{(N)}(\sigma) = P\left[\sum_{\nu=1}^{n} r_\nu \leqslant \sigma; \rho \in S_{k,\mu}^{(N)}\right], \tag{47}$$

and denote by $H_{k,\mu}(N)$ the probability that the leading index μ in $S^{(N)}$ follows the leading index k.

26. We define further

$$H_k = \lim_{N\to\infty} \frac{1}{N} L_k(N), \tag{48}$$

$$H_{k,\mu} = \lim_{N\to\infty} H_{k,\mu}(N), \tag{49}$$

$$\Phi_k(\sigma) = \lim_{N\to\infty} \Phi_k^{(N)}(\sigma), \tag{50}$$

$$\Phi_{k,\mu}(\sigma) = \lim_{N\to\infty} \Phi_{k,\mu}^{(N)}(\sigma) \tag{51}$$

if the limits (48)–(51) exist. We have then obviously by elementary probabilities

$$\Phi^{(N)}(\sigma) = \sum_{k=1}^{n} \Phi_k^{(N)}(\sigma) \frac{L_k(N)}{N+1}, \tag{52}$$

$$\Phi_k^{(N)}(\sigma) = \sum_{\mu=1}^{n} \Phi_{k,\mu}^{(N)}(\sigma) H_{k,\mu}(N). \tag{53}$$

We see that, if the limits (48), (49) and (51) exist, $\Phi_k(\sigma)$ and $\Phi(\sigma)$ also exist and we have

$$\Phi(\sigma) = \sum_{k=1}^{n} \Phi_k(\sigma) H_k, \tag{54}$$

$$\Phi_k(\sigma) = \sum_{\mu=1}^{n} \Phi_{k,\mu}(\sigma) H_{k,\mu}. \tag{55}$$

27. We define now n vector-fields F_k consisting respectively of all vectors ρ normed according to (43), for which the component r_k vanishes. Each of the fields

F_k can again be decomposed into $n-1$ vector-fields $F_{k,\mu}(k \neq \mu)$ consisting respectively of all vectors ρ normed according to (43) with $r_k = 0$, $|r_\mu| = \beta_\mu$. A vector ρ can of course belong to several of these fields.

In the fields F_k, $F_{k,\mu}$ we will now define in a suitable way the following *geometric probabilities:*

$$\Phi_k^*(\sigma) = P\left[\sum_{\nu=1}^{n} r_\nu \leqslant \sigma;\, \rho \in F_k\right], \tag{56}$$

$$\omega_{k,\mu} = P[|r_\mu| = \beta_\mu;\, \rho \in F_k], \tag{57}$$

$$\Phi^*_{k,\mu}(\sigma) = P\left[\sum_{\nu=1}^{n} r_\nu \leqslant \sigma;\, \rho \in F_{k,\mu}\right]. \tag{58}$$

Of course, to define these geometric probabilities we must use a convenient model, which presents itself here in a very natural way. If the components $r_\nu^{(\lambda)}$ of the residual vector ρ_λ are computed up to a certain number of decimal places, they are approximated by certain fractions $\dfrac{q_\nu^{(\lambda)}}{m}$, where m is a power of 10. It is then natural to assume that, if the subscripts k and μ are fixed, the integers $q_\nu^{(\lambda)}$ vary uniformly within their limits. We consider therefore $n+1$ integers

$$m,\ m_\nu \quad (\nu = 1, \cdots, n)$$

tending to infinity in such a way that we have

$$\frac{m_\nu}{m} \to \beta_\nu \quad (\nu = 1, \cdots, n).$$

We define then

$$\Phi_k^*(\sigma) = \lim_{m\to\infty} P\left[\sum_{\nu=1}^{n} q_\nu \leqslant \sigma m;\, \max_{1\leqslant\nu\leqslant n} \frac{|q_\nu|}{m_\nu} = 1,\, q_k = 0\right], \tag{59}$$

$$\omega_{k,\mu} = \lim_{m\to\infty} P\left[|q_\mu| = m_\mu;\, \max_{1\leqslant\nu\leqslant n} \frac{|q_\nu|}{m_\nu} = 1,\, q_k = 0\right], \tag{60}$$

$$\Phi^*_{k,\mu} = \lim_{m\to\infty} P\left[\sum_{\nu=1}^{n} q_\nu \leqslant \sigma m;\, |q_\mu| = m_\mu,\, q_k = 0,\, \max_{1\leqslant\nu\leqslant n} \frac{|q_\nu|}{m_\nu} = 1\right]. \tag{61}$$

28. It is easy to compute the $\omega_{k,\mu}$. Assume $k = 1$. Then the probability $\omega_{1,\mu}$ in (60) is obtained in considering the right parallelepiped $|x_\lambda| \leqslant m_\lambda$ $(\lambda = 2, \cdots, n)$, as the quotient of the area of the two faces $|x_\mu| = m_\mu$ by the complete area of all faces. In putting temporarily $m_2 \cdots m_n = M$ we obtain

$$\frac{2M/m_\mu}{2\sum_{\lambda=2}^{n} M/m_\lambda} = \frac{m/m_\mu}{\sum_{\lambda=2}^{n} m/m_\lambda},$$

and this tends to

$$\frac{1/\beta_\mu}{\sum_{\nu=1}^{n} 1/\beta_\nu - 1/\beta_1}.$$

In replacing here β_1 by β_k we obtain finally

$$\omega_{k,\mu} = \frac{1/\beta_\mu}{\sum_{\nu=1}^{n} 1/\beta_\nu - 1/\beta_k} \quad (k \neq \mu). \tag{62}$$

We write this differently in introducing the expressions

$$\sigma_\lambda = \frac{1/\beta_\lambda}{\sum_{\nu=1}^{n} 1/\beta_\nu}. \tag{63}$$

Then we have obviously

$$\sum_{\lambda=1}^{n} \sigma_\lambda = 1, \tag{64}$$

and (62) becomes

$$\frac{\sigma_\mu}{\sum_{\nu=1}^{n} \sigma_\nu - \sigma_k} = \frac{\sigma_\mu}{1 - \sigma_k}.$$

If we now define $\omega_{k,k}$ as 0, we obtain

$$\omega_{k,\mu} = \begin{cases} \dfrac{\sigma_\mu}{1 - \sigma_k} & (k \neq \mu), \\ 0 & (k = \mu). \end{cases} \tag{65}$$

The values $\Phi^*_{k,\mu}(\sigma)$ and $\Phi_k^*(\sigma)$ have already been computed in another communication [8], and we will give $\Phi^*_{k,\mu}(\sigma)$ later on. In particular we have

$$\Phi_k^*(\sigma) = \sum_{\substack{\mu=1 \\ \mu \neq k}}^{n} \omega_{k,\mu} \Phi^*_{k,\mu}(\sigma). \tag{66}$$

29. We make now two fundamental assumptions about the sequence S under consideration:

(67) *Hypothesis A.* $\Phi_{k,\mu}(\sigma) = \Phi^*_{k,\mu}(\sigma) \quad (k \neq \mu;\ k, \mu = 1, \cdots, n)$

(68) *Hypothesis B.* $H_{k,\mu}(N) \to \omega_{k,\mu} \quad (N \to \infty,\ k \neq \mu;\ k, \mu = 1, \cdots, n).$

It follows then from (55) that we have

$$\Phi_k(\sigma) = \Phi_k^*(\sigma). \tag{69}$$

In order to obtain the value of $\Phi(\sigma)$ we will now prove the existence of the limit (48).

30. We have obviously

$$\sum_{k=1}^{n} L_k(N) = N + 1, \tag{70}$$

$$L_\mu(N) = \sum_{\substack{k=1 \\ k\neq\mu}}^{n} H_{k,\mu}(N) L_k(N-1).$$

If we replace here each $L_k(N-1)$ by $L_k(N)$, the modulus of the error does not exceed $\sum_{\substack{k=1 \\ k\neq\mu}}^{n} H_{k,\mu} = 1$, and we obtain

$$\sum_{\substack{k=1 \\ k\neq\mu}}^{n} H_{k,\mu}(N) L_k(N) = L_\mu(N) + e_\mu(N), \quad |e_\mu(N)| \leqslant 1.$$

Divide this by N and put

$$\frac{L_k(N)}{N} = h_k(N). \tag{71}$$

Then we obtain

$$\sum_{\substack{k=1 \\ k\neq\mu}}^{n} H_{k,\mu}(N) h_k(N) - h_\mu(N) = \epsilon_\mu(N), \quad \epsilon_\mu(N) = O\left(\frac{1}{N}\right) \quad (\mu = 1, \cdots, n). \tag{72}$$

In the same way we obtain from (70)

$$\sum_{k=1}^{n} h_k(N) = 1 + \frac{1}{N}. \tag{73}$$

If we replace in the equations (72) and (73) the $h_\mu(N)$ by x_μ, the equations become for $N \to \infty$ by (68) and (65)

$$\sigma_\mu \sum_{\substack{k=1 \\ k\neq\mu}}^{n} \frac{x_k}{1-\sigma_k} = x_\mu \quad (\mu = 1, \cdots, n), \tag{74}$$

$$\sum_{k=1}^{n} x_k = 1. \tag{75}$$

We consider first the system (74). If we divide by σ_μ and add $\dfrac{x_\mu}{1-\sigma_\mu}$ on both sides, we obtain

$$\sum_{k=1}^{n} \frac{x_k}{1-\sigma_k} = \frac{x_\mu}{\sigma_\mu} + \frac{x_\mu}{1-\sigma_\mu} = \frac{x_\mu}{\sigma_\mu(1-\sigma_\mu)}.$$

Denoting the expression on the left by γ we get

$$x_\mu = \gamma\sigma_\mu(1-\sigma_\mu). \tag{76}$$

31. On the other hand, if we put the value (76) for $\gamma = 1$ into the system (74), the system is satisfied; we see that the rank of (74) is exactly $n-1$. We can

therefore choose $n - 1$ of the equations (74) in such a way that they form, taken together with (75), a system of the rank n. Therefore the corresponding $n - 1$ equations (72) form, taken together with the equation (73), a linear system which remains regular in the limit $N \to \infty$. Therefore its solutions $h_k(N)$ tend to the solution of the system (74), (75); and this solution is obtained from (76), if γ is chosen so that (75) is fulfilled. But then we have

$$\sum_{\mu=1}^{n} x_\mu = 1 = \gamma\left(1 - \sum_{\mu=1}^{n} \sigma_\mu^2\right),$$

$$\frac{L_k(N)}{N} \to H_k = \frac{\sigma_k(1 - \sigma_k)}{1 - \sum_{\nu=1}^{n} \sigma_\nu^2}. \tag{77}$$

If we introduce here the values (63) of the σ_ν, we obtain

$$H_k = \frac{\beta_k^{-1}(\sum_\lambda \beta_\lambda^{-1} - \beta_k^{-1})}{2 \sum_{\nu \neq \lambda} \beta_\nu^{-1}\beta_\lambda^{-1}}. \tag{78}$$

It follows now from (54), (55), (68), (65):

$$\Phi(\sigma) = \sum_{k=1}^{n} H_k \sum_{\substack{\mu=1 \\ \mu \neq k}}^{n} \omega_{k,\mu}\Phi^*_{k,\mu}(\sigma). \tag{79}$$

32. An explicit formula for $\Phi^*_{k,\mu}(\sigma)$ can be written simply by using the expression for a function $F(\sigma)$ which was defined and computed in a previous paper [8]. In replacing the integer n used in that paper by $n - 2$ (for $n \geqslant 3$) we define $F(\sigma)$ as the probability

$$F(\sigma) = P[\xi_1 + \xi_2 + \cdots + \xi_{n-2} \leqslant \sigma; \; |\xi_\nu| \leqslant \alpha_\nu \quad (\nu = 1, \cdots, n - 2)]. \tag{80}$$

The α_ν are $n - 2$ positive numbers. Then we have for $F(\sigma)$ (see formula (5) in [8]):

$$F(\sigma) = \frac{1}{2^{n-2}(n-2)!} \cdot \frac{1}{\alpha_1 \cdots \alpha_{n-2}} \prod_{\nu=1}^{n-2} (1 - S^{2\alpha_\nu})(\alpha + \sigma)_+^{n-2}, \tag{81}$$

$$\alpha = \alpha_1 + \cdots + \alpha_{n-2}.$$

Here S is the operator defined by

$$S^\eta f(\alpha) = f(\alpha - \eta).$$

The computed expression consists of 2^{n-2} terms of the form

$$\pm (\alpha - 2\alpha_{\lambda_1} - 2\alpha_{\lambda_2} - \cdots + \sigma)_+^{n-2}.$$

The subscript $+$ signifies that, if the expression within the parentheses is negative, the whole expression has to be replaced by 0, while otherwise the subscript can be dropped.

In particular we have pointed out in [8], p. 6, that $F(\sigma)$ is *strictly monotonically increasing* with σ for $-\alpha \leqslant \sigma \leqslant \alpha$ and grows from $F(-\alpha) = 0$ to $F(\alpha) = 1$.

We see that we have

$$1 > F(\sigma) > 0 \quad (|\sigma| < \alpha). \tag{82}$$

$F(\sigma)$ is further a continuous function of σ. We obtain therefore the same value of the probability, if we replace in (80) the condition $\leqslant \sigma$ by the condition $< \sigma$.

33. $\Phi^*_{k,\mu}(\sigma)$ is by definition (58) the arithmetical mean of the two probabilities

$$P\Big[\sum_{\substack{\nu=1\\ \nu\neq k,\mu}}^{n} r_\nu \leqslant \sigma - \beta_\mu;\ |r_\nu| \leqslant \beta_\nu\Big],\ P\Big[\sum_{\substack{\nu=1\\ \nu\neq k,\mu}}^{n} r_\nu \leqslant \sigma + \beta_\mu;\ |r_\nu| \leqslant \beta_\nu\Big].$$

We have therefore

$$\Phi_{k,\mu}(\sigma) = \tfrac{1}{2}[F_{k,\mu}(\sigma - \beta_\mu) + F_{k,\mu}(\sigma + \beta_\mu)]. \tag{83}$$

The function $F_{k,\mu}$ is here obtained from (81), if the $\alpha_1, \cdots, \alpha_{n-2}$ are identified with the $n - 2$ of the numbers β_ν which remain after deleting β_μ and β_k. Introducing (83) into (79) we obtain finally the complete expression of $\Phi(\sigma)$.

If we return now to the problem of computing the probability (42) and norm here the r_ν according to (43), we see that we have to compute

$$P\Big[\Big|\sum_{\nu=1}^{n} r_\nu\Big| > \beta_0,\ \max_{1\leqslant\nu\leqslant n} \frac{|r_\nu|}{\beta_\nu} = 1,\ \min_{1\leqslant\nu\leqslant n} |r_\nu| = 0\Big].$$

This is the sum of the two probabilities

$$P\Big[\sum_{\nu=1}^{n} r_\nu < -\beta_0;\ \max_{1\leqslant\nu\leqslant n} \frac{|r_\nu|}{\beta_\nu} = 1,\ \min_{1\leqslant\nu\leqslant n} |r_\nu| = 0\Big],$$

$$P\Big[\sum_{\nu=1}^{n} r_\nu > \beta_0;\ \max_{1\leqslant\nu\leqslant n} \frac{|r_\nu|}{\beta_\nu} = 1,\ \min_{1\leqslant\nu\leqslant n} |r_\nu| = 0\Big].$$

But these two probabilities have the same value, and the first of them is obviously $\Phi(-\beta_0)$. We obtain therefore for the probability (42) the expression

$$2\Phi(-\beta_0) = \sum_{k=1}^{n} H_k \sum_{\substack{\mu=1\\ \mu\neq k}}^{n} \omega_{k,\mu}[F_{k,\mu}(-\beta_0 - \beta_\mu) + F_{k,\mu}(-\beta_0 + \beta_\mu)]. \tag{84}$$

34. We will now prove that (84) is *positive* under the condition (40), that is

$$\beta_0 < \sum_{\nu=1}^{n} \beta_\nu - \min_{1\leqslant\nu\leqslant n} \beta_\nu. \tag{85}$$

Observe that all terms in (84) are non-negative. It is therefore sufficient to prove that under the condition (85) at least one of the expressions $F_{k,\mu}(-\beta_0 \pm \beta_\mu)$ is positive, that is to say, by (82), that for convenient choice of k and μ either $\beta_0 + \beta_\mu$ or $|\beta_0 - \beta_\mu|$ is less than $\sum_{\nu=1}^{n} \beta_\nu - \beta_\mu - \beta_k$. Assume now that the β_ν are ordered increasingly

$$\beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_n$$

and put

$$\beta = \beta_1 + \cdots + \beta_n.$$

Then I say that for $k = 1, \mu = 2$ we always have

$$|\beta_0 - \beta_2| < \beta - \beta_1 - \beta_2, \tag{86}$$

that is to say that

$$F_{1,2}(-\beta_0 + \beta_2) > 0.$$

To prove (86) consider the two cases $\beta_0 \geqslant \beta_2$ and $\beta_0 < \beta_2$. In the first case (86) reduces to

$$\beta_0 - \beta_2 < \beta - \beta_1 - \beta_2, \quad \beta_0 < \beta - \beta_1,$$

and this is exactly (85). In the second case (86) reduces to

$$\beta_2 - \beta_0 < \beta - \beta_1 - \beta_2, \quad \beta_0 > -(\beta - \beta_1 - 2\beta_2);$$

but here the expression on the right is not positive, as

$$(\beta_3 - \beta_2) + \beta_4 + \cdots + \beta_n \geqslant 0.$$

We have now proved that under the condition (40) and the assumptions of Nr. 29 the probability for the speeding up of Seidel's relaxation procedure in using Gauss' device is *positive*.

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Extension of Lindow's Tables for Numerical Differentiation Using Newton-Stirling and Newton-Bessel Differences

By Herbert E. Salzer and Genevieve M. Kimbro

Lindow [1] gives several short tables of coefficients for obtaining the first and second derivative at intermediate points by differentiation of either the Newton-Stirling or Newton-Bessel interpolation formula [2]. The former involves central differences $\mu\delta_0^{2m+1}$ and δ_0^{2m} which are on the line with $f(x_0)$, and the latter involves central differences $\delta_{\frac{1}{2}}^{2m+1}$ and $\mu\delta_{\frac{1}{2}}^{2m}$ which are on the line with $f\left(x_0 + \frac{h}{2}\right)$, h being the tabular interval. Lindow's tables of Newton-Stirling coefficients are intended to interpolate for $f'(x_0 + ph)$ and $f''(x_0 + ph)$ for either $0 \le p \le .25$ or for $.75 \le p \le 1.00$, in the latter case by choosing a new x_0 = original $x_0 + h$ (and corresponding central differences opposite the next entry) and a new p where now $-.25 \le$ new $p \le 0$. His tables of Newton-Bessel coefficients are intended to interpolate for $f'(x_0 + ph)$ and $f''(x_0 + ph)$ for $.25 \le p \le .75$, or in the new variable p_1 given by $p = \frac{1}{2} + p_1$, for $-.25 \le p_1 \le .25$. The reason for employing both Newton-Stirling and Newton-Bessel formulas (i.e. both tabular and mid-point central differences) is that by proper choice of p or p_1 the argument is never more than one-fourth of a tabular interval away from the central differences employed, which should yield high accuracy.

The formulas for $f'(x)$ and $f''(x)$ are of the form:

$$(1)\quad f'(x_0 \pm ph) = \frac{1}{h}\left\{\mu\delta_0 \pm p\delta_0^2 + \sum_{m=1}^{n} (A_{2m+1}(p)\mu\delta_0^{2m+1} \pm A_{2m+2}(p)\delta_0^{2m+2})\right\} + R_{1,n}$$

obtained by single differentiation of the Newton-Stirling interpolation formula, to be used for $\pm p$ ranging from $-.25$ to $.25$.

$$(2)\quad f'(x_0 + ph) \equiv f'\left(x_0 + \frac{h}{2} \pm p_1 h\right)$$

$$= \frac{1}{h}\left\{\delta_{\frac{1}{2}} \pm p_1\mu\delta_{\frac{1}{2}}^2 + \sum_{m=1}^{n} (B_{2m+1}(p_1)\delta_{\frac{1}{2}}^{2m+1} \pm B_{2m+2}(p_1)\mu\delta_{\frac{1}{2}}^{2m+2})\right\} + R_{2,n},$$

obtained by single differentiation of the Newton-Bessel interpolation formula, to be used for $p = \frac{1}{2} \pm p_1$ ranging from .25 to .75, or $\pm p_1$ ranging from $-.25$ to .25.

$$(3)\quad f''(x_0 \pm ph)$$

$$= \frac{1}{h^2}\left\{\delta_0^2 \pm p\mu\delta_0^3 + \sum_{m=1}^{n} (C_{2m+2}(p)\delta_0^{2m+2} \pm C_{2m+3}(p)\mu\delta_0^{2m+3})\right\} + R_{3,n},$$

obtained by double differentiation of the Newton-Stirling interpolation formula,

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to be used for $\pm p$ ranging from $-.25$ to $.25$.

$$(4)\quad f''(x_0 + ph) \equiv f''\left(x_0 + \frac{h}{2} \pm p_1 h\right)$$

$$= \frac{1}{h^2}\left\{\mu\delta_{\frac{1}{2}}^2 \pm p_1\delta_{\frac{1}{2}}^3 + \sum_{m=1}^{n} (D_{2m+2}(p_1)\mu\delta_{\frac{1}{2}}^{2m+2} \pm D_{2m+3}(p_1)\delta_{\frac{1}{2}}^{2m+3})\right\} + R_{4,n},$$

obtained by double differentiation of the Newton-Bessel interpolation formula, to be used for $p = \frac{1}{2} \pm p_1$ ranging from .25 to .75, or $\pm p_1$ ranging from $-.25$ to .25.

In the nth pair of terms only, the right hand term within the parentheses does not appear if the interpolation series (1), (2), (3), or (4) is not taken beyond $\mu\delta_0^{2n+1}$, $\delta_{\frac{1}{2}}^{2n+1}$, δ_0^{2n+2} or $\mu\delta_{\frac{1}{2}}^{2n+2}$ respectively. The formulas for the remainder terms $R_{i,n}$, $i = 1, 2, 3$, or 4 depend also upon the parity of the order of the last difference retained in (1), (2), (3), or (4). To find $R_{i,n}$ explicitly one may differentiate the formulas for the remainder terms in the Newton-Stirling or Newton-Bessel formula ([3], p. 100, 102), making use of the properties of divided differences under repeated differentiation ([3], p. 66–67). For (1)–(4) there will be altogether eight different formulas for $R_{i,n}$, having either two, three, four or six terms, the use of which is quite troublesome and time-consuming, since each term involves the evaluation of a polynomial of higher degree and the estimation of a higher order derivative. For most practical problems one can almost always bypass the work in using the theoretically exact expression for $R_{i,n}$, by simply observing the last term retained in any of (1)–(4) in conjunction with the rate at which the terms are falling off in magnitude.

Following are the explicit expressions for the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$, as (even or odd) polynomials in p or p_1, for $r = 3(1)10$ and $s = 4(1)10$:

$$A_3(p) = \frac{3p^2 - 1}{6},\quad A_4(p) = \frac{2p^3 - p}{12},\quad A_5(p) = \frac{5p^4 - 15p^2 + 4}{120},$$

$$A_6(p) = \frac{3p^5 - 10p^3 + 4p}{360},\quad A_7(p) = \frac{7p^6 - 70p^4 + 147p^2 - 36}{5040},$$

$$A_8(p) = \frac{2p^7 - 21p^5 + 49p^3 - 18p}{10080},$$

$$A_9(p) = \frac{9p^8 - 210p^6 + 1365p^4 - 2460p^2 + 576}{3\ 62880},$$

$$A_{10}(p) = \frac{5p^9 - 120p^7 + 819p^5 - 1640p^3 + 576p}{18\ 14400}.$$

$$B_3(p_1) = \frac{12p_1^2 - 1}{24},\qquad B_4(p_1) = \frac{4p_1^3 - 5p_1}{24},$$

$$B_5(p_1) = \frac{80p_1^4 - 120p_1^2 + 9}{1920},\quad B_6(p_1) = \frac{48p_1^5 - 280p_1^3 + 259p_1}{5760},$$

$$B_7(p_1) = \frac{448p_1^6 - 2800p_1^4 + 3108p_1^2 - 225}{3\ 22560},$$

$$B_8(p_1) = \frac{64p_1^7 - 1008p_1^5 + 3948p_1^3 - 3229p_1}{3\ 22560},$$

$$B_9(p_1) = \frac{2304p_1^8 - 37632p_1^6 + 1\ 57920p_1^4 - 1\ 54992p_1^2 + 11025}{928\ 97280},$$

$$B_{10}(p_1) = \frac{1280p_1^9 - 42240p_1^7 + 4\ 21344p_1^5 - 13\ 82480p_1^3 + 10\ 57221p_1}{4644\ 86400}.$$

$$C_4(p) = \frac{6p^2 - 1}{12}, \qquad C_5(p) = \frac{2p^3 - 3p}{12},$$

$$C_6(p) = \frac{15p^4 - 30p^2 + 4}{360}, \quad C_7(p) = \frac{3p^5 - 20p^3 + 21p}{360},$$

$$C_8(p) = \frac{14p^6 - 105p^4 + 147p^2 - 18}{10080},$$

$$C_9(p) = \frac{6p^7 - 105p^5 + 455p^3 - 410p}{30240},$$

$$C_{10}(p) = \frac{15p^8 - 280p^6 + 1365p^4 - 1640p^2 + 192}{6\ 04800}.$$

$$D_4(p_1) = \frac{12p_1^2 - 5}{24}, \qquad D_5(p_1) = \frac{4p_1^3 - 3p_1}{24},$$

$$D_6(p_1) = \frac{240p_1^4 - 840p_1^2 + 259}{5760}, \quad D_7(p_1) = \frac{336p_1^5 - 1400p_1^3 + 777p_1}{40320},$$

$$D_8(p_1) = \frac{448p_1^6 - 5040p_1^4 + 11844p_1^2 - 3229}{3\ 22560},$$

$$D_9(p_1) = \frac{576p_1^7 - 7056p_1^5 + 19740p_1^3 - 9687p_1}{29\ 03040},$$

$$D_{10}(p_1) = \frac{11520p_1^8 - 2\ 95680p_1^6 + 21\ 06720p_1^4 - 41\ 47440p_1^2 + 10\ 57221}{4644\ 86400}.$$

The purpose of these present tables is to extend Lindow's tables which go only as far as the coefficients of the 6th differences in formulas (1)–(4) above, and give only 5D up to the 4th difference coefficients and 4D for the 5th and 6th difference coefficients. These present tables give the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$ at the same interval of .01 and the same range of p or p_1 from 0 to .25, as occur in Lindow, but for every difference as far as the 10th difference inclusive, and to 10 significant figures. This represents a considerable extension of Lindow's original tables and should be useful in many calculations of the first and second derivatives at intermediate points, which arise in numerical differentiation work and in the solution of first or second order differential equations, where Lindow's tables are entirely inadequate. Anyone who has performed numerical differentiation, especially for the second derivative, is aware of the great loss in significant figures due to the power of h in the denominator, as well as the subtraction of nearly equal terms in the numerator. For functions that are not deter-

mined by either measurement, observation, experiment or approximation, but which are mathematically defined so as to be computable to any degree of precision, the only limitation to the accuracy in numerical differentiation is due to the truncating error in using a finite number of terms of the formulas (1)–(4) and the computing error due to the carrying of a fixed number of decimal places or significant figures in the computation. Thus Lindow's original tables severely limit the accuracy attainable for mathematically defined functions because of those two mentioned reasons. These present 10S tables, as far as the 10th difference, are intended primarily to reduce considerably both truncation and computational errors.

The only other conveniently available tables for numerical differentiation at intermediate points, employing central-type differences, appear to be those of Davis [4] who gives, at intervals of .01, the Newton-Stirling and Newton-Bessel coefficients, but only as far as the 5th difference and only to 5D, and the Everett [2] coefficients as far as the 6th central difference and to 10D. However, all of Davis' coefficients are for the first derivative only. Davis' 10D Everett coefficients, giving 8-point accuracy, might suffice for most problems requiring just the first derivative. But Lindow must be extended anyhow to take care of the equally important second derivative, and thus we might as well be uniform in procedure and accuracy and use Lindow's arrangement for finding also the first derivative.

The present calculation was done originally using only a desk calculator, by exact computation of the numerators in the coefficients $A_r(p)$, $B_r(p_1)$, $C_s(p)$ and $D_s(p_1)$ in (1)–(4), and rounding only after division by the denominators. Thus all tabular entries should be correct to within a half-unit in the last (tenth) significant figure. All entries on the preliminary manuscript were rechecked by Norman Levine, employing the IBM 704, using double-precision (floating point) arithmetic. An additional differencing check was performed by hand upon the entries on the preliminary manuscript. Also a functional check was performed upon every entry in the typewritten final manuscript by computing the first and second derivatives of $(1 + x)^{10}$ for $x = 0(.01).25$ and $x = .50(.01).75$.

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NEWTON-STIRLING COEFFICIENTS FOR FIRST DERIVATIVE

$$f'(x_0 \pm ph) = \frac{1}{h}\{\mu\delta_0 \pm p\delta_0^2 + A_3(p)\mu\delta_0^3 \pm A_4(p)\delta_0^4 + A_5(p)\mu\delta_0^5 \pm A_6(p)\delta_0^6 + A_7(p)\mu\delta_0^7 \pm A_8(p)\delta_0^8 + A_9(p)\mu\delta_0^9 \pm A_{10}(p)\delta_0^{10} + \cdots\}$$

p	$A_3(p)$	$A_4(p)$	$A_5(p)$	$A_6(p)$	$A_7(p)$	$A_8(p)$	$A_9(p)$	$A_{10}(p)$
.00	−0.16666 66667	0	.(1)33333 33333	0	−.(2)71428 57143	0	.(2)15873 01587	0
.01	− .16661 66667	−.(3)83316 66667*	.(1)33320 83375	.(3)11108 33342	−.(2)71399 40615	−.(4)17852 28195	.(2)15866 23715	.(5)31736 99340
.02	− .16646 66667	−.(2)16653 33333	.(1)33283 34000	.(3)22200 00267	−.(2)71311 92698	−.(4)35675 40349	.(2)15845 90549	.(5)63419 76753
.03	− .16621 66667	−.(2)24955 00000	.(1)33220 86708	.(3)33258 35358	−.(2)71166 18392	−.(4)53440 22919	.(2)15812 03443	.(5)94994 15729
.04	− .16586 66667	−.(2)33226 66667	.(1)33133 44000	.(3)44266 75200	−.(2)70962 26026	−.(4)71117 67362	.(2)15764 64654	.(4)12640 61059
.05	− .16541 66667	−.(2)41458 33333	.(1)33021 09375	.(3)55208 59375	−.(2)70700 27260	−.(4)88678 72628	.(2)15703 77337	.(4)15760 17187
.06	− .16486 66667	−.(2)49640 00000	.(1)32883 87333	.(3)66067 31467	−.(2)70380 37078	−.(3)10609 44766	.(2)15629 45548	.(4)18852 73177
.07	− .16421 66667	−.(2)57761 66667	.(1)32721 83375	.(3)76826 40058	−.(2)70002 73785	−.(3)12333 61387	.(2)15541 74242	.(4)21912 94946
.08	− .16346 66667	−.(2)65813 33333	.(1)32535 04000	.(3)87469 39733	−.(2)69567 59001	−.(3)14037 50765	.(2)15440 69266	.(4)24935 51653
.09	− .16261 66667	−.(2)73785 00000	.(1)32323 56708	.(3)97979 92075	−.(2)69075 17655	−.(3)15718 28281	.(2)15326 37362	.(4)27915 16224
.10	− .16166 66667	−.(2)81666 66667	.(1)32087 50000	.(2)10834 16667	−.(2)68525 77976	−.(3)17373 11310	.(2)15198 86161	.(4)30846 65895
.11	− .16061 66667	−.(2)89448 33333	.(1)31826 93375	.(2)11853 84209	−.(2)67919 71488	−.(3)18999 19462	.(2)15058 24178	.(4)33724 82733
.12	− .15946 66667	−.(2)97120 00000	.(1)31541 97333	.(2)12855 40693	−.(2)67257 32996	−.(3)20593 74832	.(2)14904 60813	.(4)36544 54165
.13	− .15821 66667	−.(1)10467 16667	.(1)31232 73375	.(2)13837 26078	−.(2)66539 00578	−.(3)22154 02242	.(2)14738 06340	.(4)39300 73504
.14	− .15686 66667	−.(1)11209 33333	.(1)30899 34000	.(2)14797 81520	−.(2)65765 15574	−.(3)23677 29486	.(2)14558 71910	.(4)41988 40465
.15	− .15541 66667	−.(1)11937 50000	.(1)30541 92708	.(2)15735 49479	−.(2)64936 22573	−.(3)25160 87570	.(2)14366 69537	.(4)44602 61683
.16	− .15386 66667	−.(1)12650 66667	.(1)30160 64000	.(2)16648 73813	−.(2)64052 69397	−.(3)26602 10953	.(2)14162 12101	.(4)47138 51223
.17	− .15221 66667	−.(1)13347 83333	.(1)29755 63375	.(2)17535 99881	−.(2)63115 07091	−.(3)27998 37791	.(2)13945 13335	.(4)49591 31089
.18	− .15046 66667	−.(1)14028 00000	.(1)29327 07333	.(2)18395 74640	−.(2)62123 89904	−.(3)29347 10167	.(2)13715 87824	.(4)51956 31721
.19	− .14861 66667	−.(1)14690 16667	.(1)28875 13375	.(2)19226 46749	−.(2)61079 75274	−.(3)30645 74336	.(2)13474 50996	.(4)54228 92498
.20	− .14666 66667	−.(1)15333 33333	.(1)28400 00000	.(2)20026 66667	−.(2)59983 23810	−.(3)31891 80952	.(2)13221 19111	.(4)56404 62222
.21	− .14461 66667	−.(1)15956 50000	.(1)27901 86708	.(2)20794 86751	−.(2)58834 99273	−.(3)33082 85308	.(2)12956 09261	.(4)58478 99611
.22	− .14246 66667	−.(1)16558 66667	.(1)27380 94000	.(2)21529 61360	−.(2)57635 68560	−.(3)34216 47560	.(2)12679 39357	.(4)60447 73769
.23	− .14021 66667	−.(1)17138 83333	.(1)26837 43375	.(2)22229 46953	−.(2)56386 01676	−.(3)35290 32960	.(2)12391 28121	.(4)62306 64666
.24	− .13786 66667	−.(1)17696 00000	.(1)26271 57333	.(2)22893 02187	−.(2)55086 71722	−.(3)36302 12084	.(2)12091 95078	.(4)64051 63599
.25	− .13541 66667	−.(1)18229 16667	.(1)25683 59375	.(2)23518 88021	−.(2)53738 54864	−.(3)37249 61054	.(2)11781 60546	.(4)65678 73652

* Throughout these tables the numbers in parenthesis denotes the number of 0's between the decimal point and the first significant digit.

Newton-Bessel Coefficients for First Derivative

$$f'(x_0 + ph) = f'\left(x_0 + \frac{h}{2} \pm p_1 h\right) = \frac{1}{h}\{\delta_{\frac{1}{2}} \pm p_1 \mu \delta_{\frac{1}{2}}^2 + B_3(p_1)\delta_{\frac{1}{2}}^3 \pm B_4(p_1)\mu\delta_{\frac{1}{2}}^4 + B_5(p_1)\delta_{\frac{1}{2}}^5 \pm B_6(p_1)\mu\delta_{\frac{1}{2}}^6 + B_7(p_1)\delta_{\frac{1}{2}}^7 \pm B_8(p_1)\mu\delta_{\frac{1}{2}}^8 + B_9(p_1)\delta_{\frac{1}{2}}^9 \pm B_{10}(p_1)\mu\delta_{\frac{1}{2}}^{10} + \cdots\}$$

p_1	$B_3(p_1)$	$B_4(p_1)$	$B_5(p_1)$	$B_6(p_1)$	$B_7(p_1)$	$B_8(p_1)$	$B_9(p_1)$	$B_{10}(p_1)$
.00	−.(1)41666 66667	0	.(2)46875 00000	0	−.(3)69754 46429	0	.(3)11867 94705	0
.01	−.(1)41616 66667	−.(2)20831 66667	.(2)46812 50417	.(3)44960 41675	−.(3)69658 11880	−.(3)10009 31675	.(3)11851 26451	.(4)22758 09919
.02	−.(1)41466 66667	−.(2)41653 33333	.(2)46625 06667	.(3)89891 66933	−.(3)69369 18650	−.(3)20011 29068	.(3)11801 23731	.(4)45498 34292
.03	−.(1)41216 66667	−.(2)62455 00000	.(2)46312 83750	.(2)13476 46036	−.(3)68887 97981	−.(3)29998 58274	.(3)11717 92660	.(4)68202 88662
.04	−.(1)40866 66667	−.(2)83226 66667	.(2)45876 06667	.(2)17955 00853	−.(3)68215 01927	−.(3)39963 86136	.(3)11601 43432	.(4)90853 90750
.05	−.(1)40416 66667	−.(1)10395 83333	.(2)45315 10417	.(2)22421 90104	−.(3)67351 03330	−.(3)49899 80622	.(3)11451 90302	.(3)11343 36154
.06	−.(1)39866 66667	−.(1)12464 00000	.(2)44630 40000	.(2)26874 23147	−.(3)66296 95781	−.(3)59799 11199	.(3)11269 51584	.(3)13592 42635
.07	−.(1)39216 66667	−.(1)14526 16667	.(2)43822 50417	.(2)31309 09839	−.(3)65053 93579	−.(3)69654 49207	.(3)11054 49636	.(3)15830 81596
.08	−.(1)38466 66667	−.(1)16581 33333	.(2)42892 06667	.(2)35723 60640	−.(3)63623 31677	−.(3)79458 68231	.(3)10807 10839	.(3)18056 76765
.09	−.(1)37616 66667	−.(1)18628 50000	.(2)41839 83750	.(2)40114 86707	−.(3)62006 65610	−.(3)89204 44478	.(3)10527 65586	.(3)20268 52628
.10	−.(1)36666 66667	−.(1)20666 66667	.(2)40666 66667	.(2)44480 00000	−.(3)60205 71429	−.(3)98884 57143	.(3)10216 48254	.(3)22464 34540
.11	−.(1)35616 66667	−.(1)22694 83333	.(2)39373 50417	.(2)48816 13376	−.(3)58222 45608	−.(2)10849 18879	.(4)98739 71820	.(3)24642 48828
.12	−.(1)34466 66667	−.(1)24712 00000	.(2)37961 40000	.(2)53120 40693	−.(3)56059 04957	−.(2)11801 92570	.(4)95005 46433	.(3)26801 22902
.13	−.(1)33216 66667	−.(1)26717 16667	.(2)36431 50417	.(2)57389 96911	−.(3)53717 86508	−.(2)12745 95828	.(4)90966 68144	.(3)28938 85357
.14	−.(1)31866 66667	−.(1)28709 33333	.(2)34785 06667	.(2)61621 98187	−.(3)51201 47407	−.(2)13680 58139	.(4)86628 37422	.(3)31053 66083
.15	−.(1)30416 66667	−.(1)30687 50000	.(2)33023 43750	.(2)65813 61979	−.(3)48512 64788	−.(2)14605 09473	.(4)81995 93076	.(3)33143 96367
.16	−.(1)28866 66667	−.(1)32650 66667	.(2)31148 06667	.(2)69962 07147	−.(3)45654 35634	−.(2)15518 80322	.(4)77075 11867	.(3)35208 09000
.17	−.(1)27216 66667	−.(1)34597 83333	.(2)29160 50417	.(2)74064 54047	−.(3)42629 76636	−.(2)16421 01733	.(4)71872 08092	.(3)37244 38378
.18	−.(1)25466 66667	−.(1)36528 00000	.(2)27062 40000	.(2)78118 24640	−.(3)39442 24037	−.(2)17311 05347	.(4)66393 33132	.(3)39251 20609
.19	−.(1)23616 66667	−.(1)38440 16667	.(2)24855 50417	.(2)82120 42582	−.(3)36095 33465	−.(2)18188 23433	.(4)60645 74986	.(3)41226 93613
.20	−.(1)21666 66667	−.(1)40333 33333	.(2)22541 66667	.(2)86068 33333	−.(3)32592 79762	−.(2)19051 88929	.(4)54636 57763	.(3)43169 97227
.21	−.(1)19616 66667	−.(1)42206 50000	.(2)20122 83750	.(2)89959 24251	−.(3)28938 56795	−.(2)19901 35468	.(4)48373 41157	.(3)45078 73301
.22	−.(1)17466 66667	−.(1)44058 66667	.(2)17601 06667	.(2)93790 44693	−.(3)25136 77263	−.(2)20735 97426	.(4)41864 19889	.(3)46951 65802
.23	−.(1)15216 66667	−.(1)45888 83333	.(2)14978 50417	.(2)97559 26119	−.(3)21191 72493	−.(2)21555 09946	.(4)35117 23124	.(3)48787 20914
.24	−.(1)12866 66667	−.(1)47696 00000	.(2)12257 40000	.(1)10126 30219	−.(3)17107 92221	−.(2)22358 08982	.(4)28141 13861	.(3)50583 87133
.25	−.(1)10416 66667	−.(1)49479 16667	.(3)94401 04167	.(1)10489 90885	−.(3)12890 04371	−.(2)23144 31327	.(4)20944 88295	.(3)52340 15366

NEWTON-STIRLING COEFFICIENTS FOR SECOND DERIVATIVE

$$f''(x_0 \pm ph) = \frac{1}{h^2}\{\delta_0^2 \pm p\mu\delta_0^3 + C_4(p)\delta_0^4 \pm C_5(p)\mu\delta_0^5 + C_6(p)\delta_0^6 \pm C_7(p)\mu\delta_0^7 + C_8(p)\delta_0^8 \pm C_9(p)\mu\delta_0^9 + C_{10}(p)\delta_0^{10} \pm \cdots\}$$

p	$C_4(p)$	$C_5(p)$	$C_6(p)$	$C_7(p)$	$C_8(p)$	$C_9(p)$	$C_{10}(p)$
.00	−.(1)83333 33333	0	.(1)11111 11111	0	−.(2)17857 14286	0	.(3)31746 03175
.01	−.(1)83283 33333	−.(2)24998 33333	.(1)11102 77819	.(3)58327 77786	−.(2)17842 56057	−.(3)13556 69646	.(3)31718 91760
.02	−.(1)83133 33333	−.(2)49986 66667	.(1)11077 78444	.(2)11662 22249	−.(2)17798 82619	−.(3)27104 36619	.(3)31637 60225
.03	−.(1)82883 33333	−.(2)74955 00000	.(1)11036 14486	.(2)17485 00202	−.(2)17725 97722	−.(3)40633 98661	.(3)31502 16691
.04	−.(1)82533 33333	−.(2)99893 33333	.(1)10977 88444	.(2)23297 78631	−.(2)17624 07613	−.(3)54136 54349	.(3)31312 74690
.05	−.(1)82083 33333	−.(1)12479 16667	.(1)10903 03819	.(2)29097 24826	−.(2)17493 21035	−.(3)67603 03508	.(3)31069 53156
.06	−.(1)81533 33333	−.(1)14964 00000	.(1)10811 65111	.(2)34880 06480	−.(2)17333 49221	−.(3)81024 47629	.(3)30772 76411
.07	−.(1)80883 33333	−.(1)17442 83333	.(1)10703 77819	.(2)40642 91784	−.(2)17145 05893	−.(3)94391 90286	.(3)30422 74152
.08	−.(1)80133 33333	−.(1)19914 66667	.(1)10579 48444	.(2)46382 49529	−.(2)16928 07255	−.(2)10769 63755	.(3)30019 81432
.09	−.(1)79283 33333	−.(1)22378 50000	.(1)10438 84486	.(2)52095 49207	−.(2)16682 71985	−.(2)12092 89839	.(3)29564 38639
.10	−.(1)78333 33333	−.(1)24833 33333	.(1)10281 94444	.(2)57778 61111	−.(2)16409 21230	−.(2)13408 08512	.(3)29056 91471
.11	−.(1)77283 33333	−.(1)27278 16667	.(1)10108 87819	.(2)63428 56431	−.(2)16107 78596	−.(2)14714 31378	.(3)28497 90905
.12	−.(1)76133 33333	−.(1)29712 00000	.(2)99197 51111	.(2)69042 07360	−.(2)15778 70139	−.(2)16010 70456	.(3)27887 93171
.13	−.(1)74883 33333	−.(1)32133 83333	.(2)97146 78194	.(2)74615 87189	−.(2)15422 24353	−.(2)17296 38221	.(3)27227 59712
.14	−.(1)73533 33333	−.(1)34542 66667	.(2)94937 84444	.(2)80146 70409	−.(2)15038 72161	−.(2)18570 47646	.(3)26517 57149
.15	−.(1)72083 33333	−.(1)36937 50000	.(2)92572 04861	.(2)85631 32812	−.(2)14628 46903	−.(2)19832 12242	.(3)25758 57237
.16	−.(1)70533 33333	−.(1)39317 33333	.(2)90050 84444	.(2)91066 51591	−.(2)14191 84317	−.(2)21080 46096	.(3)24951 36826
.17	−.(1)68883 33333	−.(1)41681 16667	.(2)87375 78194	.(2)96449 05436	−.(2)13729 22532	−.(2)22312 63918	.(3)24096 77804
.18	−.(1)67133 33333	−.(1)44028 00000	.(2)84548 51111	.(1)10177 57464	−.(2)13241 02047	−.(2)23533 81076	.(3)23195 67055
.19	−.(1)65283 33333	−.(1)46356 83333	.(2)81570 78194	.(1)10704 34119	−.(2)12727 65715	−.(2)24737 13638	.(3)22248 96400
.20	−.(1)63333 33333	−.(1)48666 66667	.(2)78444 44444	.(1)11224 88889	−.(2)12189 58730	−.(2)25923 78413	.(3)21257 62540
.21	−.(1)61283 33333	−.(1)50956 50000	.(2)75171 44861	.(1)11738 90342	−.(2)11627 28604	−.(2)27092 92989	.(3)20222 66995
.22	−.(1)59133 33333	−.(1)53225 33333	.(2)71753 84444	.(1)12246 07247	−.(2)11041 25147	−.(2)28243 75776	.(3)19145 16044
.23	−.(1)56883 33333	−.(1)55472 16667	.(2)68193 78194	.(1)12746 08584	−.(2)10432 00451	−.(2)29375 46042	.(3)18026 20653
.24	−.(1)54533 33333	−.(1)57696 00000	.(2)64493 51111	.(1)13238 63552	−.(3)98000 88649	−.(2)30487 23954	.(3)16866 96407
.25	−.(1)52083 33333	−.(1)59895 83333	.(2)60655 38194	.(1)13723 41580	−.(3)91460 69723	−.(2)31578 30617	.(3)15668 63438

NEWTON-BESSEL COEFFICIENTS FOR SECOND DERIVATIVE

$$f''(x_0 + ph) = f''\left(x_0 + \frac{h}{2} \pm p_1 h\right) = \frac{1}{h^2}\{\mu\delta_{1/2}^2 \pm p_1\delta_{1/2}^3 + D_4(p_1)\mu\delta_{1/2}^4 \pm D_5(p_1)\delta_{1/2}^5 + D_6(p_1)\mu\delta_{1/2}^6 \pm D_7(p_1)\delta_{1/2}^7 + D_8(p_1)\mu\delta_{1/2}^8 \pm D_9(p_1)\delta_{1/2}^9 + D_{10}(p_1)\mu\delta_{1/2}^{10} \pm \cdots\}$$

p_1	$D_4(p_1)$	$D_5(p_1)$	$D_6(p_1)$	$D_7(p_1)$	$D_8(p_1)$	$D_9(p_1)$	$D_{10}(p_1)$
.00	−0.20833 33333	0	.(1)44965 27778	0	−.(1)10010 54067	0	.(2)22761 07546
.01	− .20828 33333	−.(2)12498 33333	.(1)44950 69486	.(3)19267 36119	−.(1)10006 86896	−.(4)33361 66939	.(2)22752 14682
.02	− .20813 33333	−.(2)24986 66667	.(1)44906 95111	.(3)38513 89156	−.(2)99958 55675	−.(4)66682 54746	.(2)22725 36636
.03	− .20788 33333	−.(2)37455 00000	.(1)44834 06153	.(3)57718 77025	−.(2)99775 06455	−.(4)99921 87205	.(2)22680 75040
.04	− .20753 33333	−.(2)49893 33333	.(1)44732 05111	.(3)76861 19644	−.(2)99518 30669	−.(3)13303 89393	.(2)22618 32613
.05	− .20708 33333	−.(2)62291 66667	.(1)44600 95486	.(3)95920 39931	−.(2)99188 41434	−.(3)16599 31329	.(2)22538 13163
.06	− .20653 33333	−.(2)74640 00000	.(1)44440 81778	.(2)11487 56480	−.(2)98785 55610	−.(3)19874 39529	.(2)22440 21580
.07	− .20588 33333	−.(2)86928 33333	.(1)44251 69486	.(2)13370 62617	−.(2)98309 93792	−.(3)23125 10452	.(2)22324 63839
.08	− .20513 33333	−.(2)99146 66667	.(1)44033 65111	.(2)15239 16196	−.(2)97761 80311	−.(3)26347 42301	.(2)22191 46993
.09	− .20428 33333	−.(1)11128 50000	.(1)43786 76153	.(2)17091 11707	−.(2)97141 43218	−.(3)29537 35317	.(2)22040 79174
.10	− .20333 33333	−.(1)12333 33333	.(1)43511 11111	.(2)18924 44444	−.(2)96449 14286	−.(3)32690 92063	.(2)21872 69587
.11	− .20228 33333	−.(1)13528 16667	.(1)43206 79486	.(2)20737 10598	−.(2)95685 28995	−.(3)35804 17718	.(2)21687 28508
.12	− .20113 33333	−.(1)14712 00000	.(1)42873 91778	.(2)22527 07360	−.(2)94850 26527	−.(3)38873 20359	.(2)21484 67277
.13	− .19988 33333	−.(1)15883 83333	.(1)42512 59486	.(2)24292 33022	−.(2)93944 49752	−.(3)41894 11248	.(2)21264 98296
.14	− .19853 33333	−.(1)17042 66667	.(1)42122 95111	.(2)26030 87076	−.(2)92968 45217	−.(3)44863 05120	.(2)21028 35021
.15	− .19708 33333	−.(1)18187 50000	.(1)41705 12153	.(2)27740 70312	−.(2)91922 63136	−.(3)47776 20463	.(2)20774 91961
.16	− .19553 33333	−.(1)19317 33333	.(1)41259 25111	.(2)29419 84924	−.(2)90807 57373	−.(3)50629 79804	.(2)20504 84665
.17	− .19388 33333	−.(1)20431 16667	.(1)40785 49486	.(2)31066 34603	−.(2)89623 85431	−.(3)53420 09988	.(2)20218 29721
.18	− .19213 33333	−.(1)21528 00000	.(1)40284 01778	.(2)32678 24640	−.(2)88372 08435	−.(3)56143 42458	.(2)19915 44748
.19	− .19028 33333	−.(1)22606 83333	.(1)39754 99486	.(2)34253 62027	−.(2)87052 91114	−.(3)58796 13532	.(2)19596 48385
.20	− .18833 33333	−.(1)23666 66667	.(1)39198 61111	.(2)35790 55556	−.(2)85667 01786	−.(3)61374 64683	.(2)19261 60288
.21	− .18628 33333	−.(1)24706 50000	.(1)38615 06153	.(2)37287 15917	−.(2)84215 12336	−.(3)63875 42808	.(2)18911 01118
.22	− .18413 33333	−.(1)25725 33333	.(1)38004 55111	.(2)38741 55804	−.(2)82697 98203	−.(3)66295 00507	.(2)18544 92534
.23	− .18188 33333	−.(1)26722 16667	.(1)37367 29486	.(2)40151 90008	−.(2)81116 38350	−.(3)68629 96348	.(2)18163 57183
.24	− .17953 33333	−.(1)27696 00000	.(1)36703 51778	.(2)41516 35520	−.(2)79471 15254	−.(3)70876 95138	.(2)17767 18691
.25	− .17708 33333	−.(1)28645 83333	.(1)36013 45486	.(2)42833 11632	−.(2)77763 14872	−.(3)73032 68190	.(2)17356 01652

TECHNICAL NOTES AND SHORT PAPERS

Table of Integers Not Exceeding 10 00000 That Are Not Expressible as the Sum of Four Tetrahedral Numbers

By Herbert E. Salzer and Norman Levine

For the past 21 years one of the authors has been concerned with empirical theorems expressing certain classes of positive integers as the sum of four tetrahedral numbers, or "tetrahedrals" $T_n \equiv n(n+1)(n+2)/6$, $n \geq 0$, the results being contained in short notes and abstracts, as well as unpublished tables [1]–[9]. We use Σ_p to denote a sum of p T_n's. Among the more interesting past findings were: every square $\leq$ 10 00000 a Σ_4 [7], [8], every $T_n \leq$ 10 00000 a Σ_4 other than the trivial decomposition $T_n = T_n + 0 + 0 + 0$ [3], [4], every multiple of 5 $\leq$ 1 00000 a Σ_4 [9] (see also [11]), and verification of Pollock's conjecture that every integer is a Σ_5 for the first 20000 integers [5], [10]. Incidentally all investigations in [1]–[9] were done by hand, employing at the most a desk calculator.

The exceptional numbers, which by definition are those not expressible as Σ_4, were tabulated previously only up to 2000. But even that far interesting features turned up, such as 1314 being the only exceptional number ending in 4, and the very few ending in 1 or 9 [4]. Then for numbers ending in 6 and $\leq$ 20000, 6186 turned out to be the only exceptional one [6]. Also there was no striking difference between the density of exceptional numbers in the first thousand and in the second thousand brackets, decreasing from around $4\frac{1}{2}\%$ to 3%, so that it was interesting to speculate upon the approximate density in the neighborhood of say 10 00000. Then finally, the verification of the conjectures that every integer is a Σ_5 and that every integer $m = 5r$ is a Σ_4 for the first 20000 cases, while every $10r + 6$, $r = 0, 1, 2, \cdots, 617$, is a Σ_4 until 6186, posed the question as to whether an exception might occur to either of the former two empirical theorems even after verification in those first 20000 cases. Thus it was felt that tabulation of the exceptional numbers $\leq$ 10 00000 would afford a much clearer picture as to their distribution and density, as well as stronger evidence for the truth of Pollock's conjecture and the author's (and Richmond's [11]) conjecture that every $m = 5r$ is a Σ_4.

This table of exceptional numbers $\leq$ 10 43999 presents a great surprise in its picture of their distribution which is entirely different from that envisioned from those $\leq$ 2000 [4]. Most strikingly unexpected is the decrease in the density of exceptional numbers from several percent in the neighborhood of 2000 to what appears to be practically zero near 10 00000. The scarcity of exceptional numbers in the higher ranges is in accordance with Hua's result that "almost all" positive integers are expressible as a Σ_4 [15], [16]. But even more, this table shows that the likelihood of a given number m being exceptional falls off so rapidly with increasing m that it appears to be a plausible conjecture that there might be some m_0 sufficiently large such that *every* $m > m_0$ is a Σ_4 (a conjecture unlikely to suggest itself from the exceptional numbers among only the first few thousand).

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Table of Exceptional Numbers $\leq$ 10 00000

17	1227	3183	9772	29157
27	1233	3218	9973	29487
33	1243	3263	10397	29938
52	1314	3463	10467	30298
73	1382	3512	10532	31973
82	1402	3887	10633	33183
83	1468	4003	10852	36262
103	1478	4307	11237	36913
107	1513	4317	11302	37798
137	1523	4563	11737	38453
153	1578	4832	11962	38707
162	1612	4923	12247	38807
217	1622	5013	12547	39693
219	1658	5142	12722	39913
227	1678	5238	12777	41278
237	1693	5283	12843	41322
247	1731	5483	12858	41433
258	1738	5508	13127	44833
268	1742	5538	13393	47627
271	1758	5563	13822	48043
282	1767	5618	14492	56467
283	1803	5647	15122	56842
302	1858	5707	15483	58613
303	1907	6022	15867	59077
313	1923	6057	16097	62158
358	1933	6067	16538	64752
383	2037	6186	16637	65253
432	2053	6213	16742	65567
437	2172	6263	17253	71157
443	2198	6343	17683	74687
447	2217	6462	17813	78003
502	2218	6863	17893	78787
548	2251	7067	18573	83603
557	2253	7278	18782	84023
558	2327	7377	19168	85993
647	2372	7387	19277	91128
662	2382	7423	20918	1 06277
667	2417	7497	21523	1 13062
709	2437	7542	22618	1 34038
713	2457	7662	22657	1 48437
718	2537	7793	23677	3 43867
722	2538	7873	24237	
842	2578	8223	24317	
863	2687	8307	24338	
898	2818	8322	25447	
953	2858	8973	25723	
1007	2898	9063	26007	
1117	2973	9488	27858	
1118	3138	9687	28617	
1153	3142	9753	28847	

A quick glance over this table is sufficient to verify Pollock's conjecture for each of the first million integers. Furthermore, from the absence of any exceptional numbers in the range 3 43867 to 10 00000 and from an upper bound to the magnitude of the first differences of tetrahedral numbers $\leq$ 2500 00000, it is easily shown that for any number m between 10 00000 and 2500 00000, it is always possible to find a T_n such that 3 43867 $< m - T_n <$ 10 00000. Thus *every* $m \leq$ 2500 00000 *is a* Σ_5. (This fact and the extreme rarity of large exceptional numbers makes the empirical theorem of Pollock that every integer is an Σ_5 as overwhelmingly certain as one can be short of actual proof.) This table verifies that every $m = 5r$ is a Σ_4 for the first 2 00000 values of r, making it also a very plausible empirical theorem. In the class of the first 1 00000 values of both $m = 10r + 4$ and $m = 10r + 6$, 1314 and 6186 respectively are the sole exceptional numbers. Among the first 1 00000 values of $m = 10r + 1$, only 271, 1731 and 2251 are exceptional. Among the first 1 00000 values of $m = 10r + 9$, only 219 and 709 are exceptional. All exceptional numbers m, where 6186 $< m <$ 10 00000, are here seen to be of the form $10r + 2$, $10r + 3$, $10r + 7$ or $10r + 8$, and that appears to be another plausible conjecture for every exceptional $m >$ 6186.

The method of computation was to obtain $\binom{i+2}{3}+\binom{j+2}{3}+\binom{k+2}{3}+\binom{l+2}{3}$ for every $i, j, k, l \geq 0$ until one found for every $m \leq$ 10 43999 either a representation as a Σ_4 or that that m was exceptional. The calculation was begun upon the Univac Scientific Computer (ERA 1103) at the Convair Digital Computing Laboratory, the initial part re-run, and then those results were checked and continued upon the IBM 704 Digital Computer. At the start upon the ERA 1103, 500 words of high speed storage with 36 binary bits in each word permitted the investigation of 18000 numbers at a time. Then the IBM 704 had at its disposal 3000 words of high speed memory, each of the 36 binary bits in a word representing a number, so that 1 08000 numbers could be investigated at one time. The 1 08000 binary bits were filled with 1's at the start, and a 0 introduced into the binary position of the word which represented a non-exceptional m. After all possible combinations of i, j, k, l had been exhausted, each of the 3000 words was searched for binary bits that remained 1 and the exceptional numbers m corresponding to those bits were printed out. In choosing combinations of i, j, k, l, repetitions due to symmetry were avoided, as well as combinations yielding an m that was either too large or too small for the group of 1 08000 numbers under consideration.

Those interested in actual mathematical proofs (which appear to be rather involved) may consult Dickson [12], [13] for earlier work, and Watson [14] for the sharpest results to date. It is rather amazing that the proved p in $m = \Sigma_p$ for every m is $p = 8$, and no better than $p = 8$ for arbitrarily large m, while the actual p (according to the evidence in this table) may be only 5 for every m, and 4 for sufficiently large m, less by 3 and 4 respectively. Considering the difficulty of the existing proof for $p = 8$ [14], one may well wonder, should $p = 5$ or $p = 4$ be the truly minimum values, for every m, and m sufficiently large, respectively, how long the world must wait and how difficult and sharp the mathematical

tools must be, until the desired proofs would be found.

References [15] and [16] were called to the author's attention by K. A. Hirsch.

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2. ——, "Table of first two hundred squares expressed as a sum of four tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 49, 1943, p. 688.
3. ——, "New tables and facts involving sums of four tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 50, 1944, p. 55.
4. ——, "On numbers expressible as the sum of four tetrahedral numbers," London Math. Soc., *Jn.*, v. 20, 1945, p. 3–4. This note includes a table of the exceptional numbers ≤ 1000 (erratum: omission of 107). That table is continued up to 2000 in *MTAC*, v. 3, 1949, p. 423–424.
5. ——, "Further empirical results on tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 52, 1946, p. 420.
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8. ——, "Representation table for squares as sums of four tetrahedral numbers," *MTAC*, v. 3, 1948, p. 316.
9. ——, "Verification of the first twenty thousand cases of an empirical theorem with the aid of a device for mass computation," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 41.
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11. H. W. Richmond, "Notes on a problem of the 'Waring' type," London Math. Soc., *Jn.*, v. 19, 1944, p. 38–41.
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12. L. E. Dickson, *History of the Theory of Numbers*, v. 2, *Diophantine Analysis*, Carnegie Institute of Washington, publication no. 256, v. II (reprinted by G. E. Stechert, N. Y., 1934), p. iv–v and Chapter I, especially p. 4, 7, 22–23, 25, 39.
13. L. E. Dickson, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939, Chap. VII, "Sums of nine values of a cubic function," p. 130–146. This contains the proof of the theorem that every integer is the sum of nine tetrahedrals, with further references to the most advanced work up to that time, that of R. D. James and L. K. Hua on the theorem that every sufficiently large integer is the sum of eight tetrahedrals.
14. G. L. Watson, "Sums of eight values of a cubic polynomial," London Math. Soc., *Jn.*, v. 27, 1952, p. 217–224. This includes the proof of the theorem that every integer is the sum of eight tetrahedrals.
15. L. K. Hua, "Sur le problème de Waring relatif à un polynome du troisième degré," *Comptes Rendus*, Academie des Sciences, Paris, v. 210, 1940, p. 650–652.
16. L. K. Hua, "On Waring's problem with cubic polynomial summands," *Sci. Report Nat. Tsing Hua Univ.*, (A) v. 4, 1940, p. 55–83.

GROUPS OF PRIMES HAVING MAXIMUM DENSITY

By John Leech

The following lists give groups of six or more primes which minimize the difference between first and last, the lists being complete for the range 50 to 100 00000. Four numbers out of nine can be prime, such as 191, 193, 197, 199. There are 897 such groups of four in the range. Five numbers out of thirteen can be prime; there are 318 such groups in the range. Six numbers out of seventeen can be prime, such as 97, 101, 103, 107, 109, 113; there are seventeen such groups in the range, centered on:

Received February 7, 1955. Due to misfiling in the *MTAC* office, this paper is appearing later than was scheduled; see *MTAC*, Review 110, v. 11, 1957, p. 274. Some of the results have meanwhile appeared in "On a generalization of the prime pair problem," by Herschel F. Smith, *MTAC*, v. 11, 1957, p. 274.

105	1091265	2839935	6503595	8741145
16065	1615845	3243345	7187775	
19425	1954365	3400215	7641375	
43785	2822715	6005895	8062005	

Seven numbers out of 21 can be prime, eight out of 27 and nine out of 31. These all occur and are listed below. It is possible for ten numbers out of 33 to be prime and eleven out of 37, but these do not occur in the range; included in the list are such groups of nine out of 33 and ten out of up to 37 as occur in the range.

	1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	7	8	9	10
																						out of		
1271	31	19	—	*	*	—	*	—	—	*	*	—	—	*	—	*	*	—	*	7	—	27	31	
5621	7	*	—	17	13	—	43	—	—	*	*	—	—	*	—	*	*	—	*	*	21			
88781	7	47	—	19	*	—	*	—	—	*	*	—	—	*	—	*	*	—	*	*	21	27	31	
113141	7	*	—	*	*	—	*	—	—	*	*	—	—	*	—	*	*	—	*	11	—	27	31	35
165701	*	*	—	*	*	—	*	—	—	*	*	—	—	103	—	53	17	—	11	7	21			
284711	7	11	—	23	101	—	*	—	—	*	*	—	—	*	—	*	*	—	*	*	21	27		
416381	7	11	—	*	*	—	*	—	—	*	*	—	—	*	—	29	*	—	*	*	—	—	33	
626591	7	11	—	*	*	—	17	—	—	*	*	—	—	*	—	*	*	—	*	*	21	—	33	
855701	7	19	—	149	*	—	*	—	—	*	*	—	—	*	—	*	*	—	*	*	21	27	31	
1068701	*	*	—	*	*	—	*	—	—	*	*	—	—	11	—	19	47	—	29	7	21			
1146761	7	*	—	43	13	—	*	—	—	*	*	—	—	*	—	*	*	—	*	*	21	27		
2580641	7	13	—	*	*	—	*	—	—	*	*	—	—	*	—	*	*	—	11	457	—	27		
6560981	7	11	—	*	29	—	*	—	—	*	*	—	—	*	—	*	*	—	*	*	21	27	33	
6937931	7	*	—	*	*	—	*	—	—	*	*	—	—	13	—	*	*	—	*	*	—	—	33	37
7540421	7	11	—	1879	13	—	41	—	—	*	*	—	—	*	—	*	*	—	*	*	21			
8573411	7	*	—	191	*	—	17	—	—	*	*	—	—	*	—	*	*	—	*	*	21			

In each line, the number in the left hand column is the first number for the line, an asterisk indicates a prime, a dash a multiple of 3 or 5 and a number the least factor of any other composite. There are thus seen to be eleven groups of seven primes in 21 numbers, eight of eight primes in 27, five of nine primes in 31 (two in the line beginning with 113141) and four more of nine in 33, and one group of ten in 35 and one of ten in 37. The range 1 to 50 is excluded as being altogether exceptional. A list of the groups of four and five has been deposited in the *UMT* file of *MTAC* (see Review **110**, *MTAC*, v. 11, 1957, p. 274).

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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

60[**A, B, C, D, H, I, K, L, M, P, R, V, X, Z**].—George E. Forsythe, *Bibliography of Russian Mathematics Books*, Chelsea Publishing Co., New York, 1956, 106 p., 20 cm. Price $3.95.

Professor Forsythe precedes his very useful bibliography of Russian mathematical books with an informative introduction which contains, among others, a complete review of the book. To quote Prof. Forsythe:

"The subject matter of the books listed is mathematics, pure and applied, including tables beyond the most elementary, but excluding descriptive geometry. There are a few titles on quantum mechanics and other branches of mathematical physics, and more on mechanics, mathematical machines and nomography, but

these topics are far from completely covered. For textbooks, the subject matter is more advanced than calculus.

The list is confined to books, excluding dissertations. Because of their mathematical interest, most of the serial monographs (Trudy) of the Steklov Mathematical Institute have also been included. The books in the list have been published (or reprinted) in Russian or Ukrainian since 1930. Translations into Russian are omitted."

The books listed cover a wide range of topics and are in very many cases written by some of the very best research men in Russia. Thus there is every reason for supporting Prof. Forsythe's thesis that these books "would be of the greatest value to mathematicians and students." If, as Prof. Forsythe seems to think, ignorance of the existence of these fine (and cheap) books and ignorance of how to obtain them are the major obstructions to their popularity, then the present bibliography ought to result in a run on Soviet mathematical books and periodicals.

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61[A, B, C, D, E, F, K, L, M, N].—*CRC Standard Mathematical Tables*, Tenth & Eleventh Editions (Formerly Mathematical Tables from Handbook of Chemistry and Physics), Chemical Rubber Publishing Company, Cleveland, Ohio, 1957, ix + 480 p., 19.5 cm. Price $3.00.

The section on Mathematical Tables from the Handbook of Chemistry and Physics considerably enlarged and greatly improved both in content and in the format of its tables made its first appearance in January 1954 as the Tenth Edition of the C. R. C. Standard Mathematical Tables.

Among the major improvements in the Tenth Edition are the following:

1. A new section on vector analysis
2. A table of Laplace transforms
3. A table of Bessel Functions of orders zero and one, Hyperbolic Bessel Functions and Spherical Bessel Functions
4. A table of sine, cosine and exponential integrals
5. A table of Elliptic Integrals of the First and Second Kinds
6. Extension of the tables of the hyperbolic functions sinh x, cosh x, and tanh x to $x = 10$
7. Extension of the table of the exponential function to $x = 10$
8. The revision and enlargement of the table of integrals to include an additional ninety-four formulas
9. A table of χ^2
10. A table of F and t for 1% and 5% distributions
11. Commissioners 1941 Standard Ordinary Mortality Table with its auxiliary table of the commutation symbols at $2\frac{1}{2}$%.

In addition to the above, the following are included in the Eleventh Edition:

1. An enlarged section on Fourier series with special power series, Bernoulli numbers and finite sine and cosine transforms
2. A new section on partial fractions with examples
3. A table of multiples of $\pi/2$
4. A table of sums of the powers of natural numbers
5. A method for the tenfold extension of the tables of Factors and Primes
6. A table of particular solutions to ordinary and partial differential equations with constant coefficients.

These and other minor changes tend to make this volume more legible and more serviceable to all.

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62[B].—TOKYO NUMERICAL COMPUTATION BUREAU, *Report No. 10. Table of Square Roots of Complex Numbers*, 1956, i + 23 p., 25 cm. Not for sale.

This table, with explanation by T. Sasaki, lists the real and imaginary parts of the square roots of $1 + ix$ and $x + i$ to $11D$ for $x = 0(.002)1$. Second differences are given. Legibility is only moderate, but the table may be useful, because existing tables for the range $0 \leq x \leq 1$ give fewer arguments and far fewer decimals; see Fletcher, Miller, and Rosenhead *Index* [1].

A. F.

1. A. FLETCHER, J. C. P. MILLER, & L. ROSENHEAD, *An Index of Mathematical Tables*, Scientific Computing Service Limited, London, 1946, p. 33 (*MTAC*, Review 233, v. 2, 1946–47, p. 13–18).

63[D, L].—G. E. REYNOLDS, *Table of* $(\sin x)/x$, Antenna Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Bedford, Mass., AFCRC-TR-57-103, ASTIA Document No. AD117063, March 1957, iii + 5 + 200 p., 26 cm.

This work was prompted by the frequent appearance of the function $(\sin x)/x$ in the mathematics of antenna design. Values are given to 8D for $x = 0(.001)49.999$. The table is machine printed in block form, the digit in the third decimal place of x being given as the top argument, and the earlier digits as left (and right) argument. There are no differences. The table was printed from punched cards produced by Datamatic Corporation of Newton Highlands, Mass. and the AFCRC Statistical Services Division.

As far as the reviewer's information goes, there are only two other tables which, although slighter on the whole, complement this new work to a significant extent. A manuscript table by C. Blanc [1] gives $(\sin x)/x$, as well as its even derivatives to order 16, to 10D for $x = 0(.01)4$, and is worth mention in the present context because of its ten decimal places. A well-known volume by K. Hayashi [2] includes a table of $(\sin x)/x$ to 8D for $x = 0(.01)10(.1)20(1)100$, and is worth mention because of its values for $x = 50(1)100$ and its general relevance. Hayashi's table, which contains errors, is very suitable for comparison with the new table, since both are strictly eight-place. The reviewer compared

some scores of corrected Hayashi values without finding any error in those of Reynolds.

A. F.

1. C. BLANC, see *MTAC*, v. 7, 1953, p. 188 (*UMT* 160).
2. K. HAYASHI, *Tafelu der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 30. See *MTAC*, v. 1, 1943–45, Review 12, p. 4.

64[D].—CLOVIS FAUCHER, *Tables trigonometriques contenant les valeurs naturelles des sinus et des cosinus de centigrade en centigrade du quadrant avec dix decimales*, Gauthier-Villars, Paris, 1957, 51 p., 27 cm. Price 500 francs.

These tables, of $\sin x$ and $\cos x$ to 10D for $x = 0(0^g.01)50^g$ with first differences, have been photo-offset from manuscript. The author states (in effect) that the values were derived by chopping the last two figures of 12D values calculated by quadratic interpolation in the fundamental tables of Andoyer. A dot appended after the 10th digit indicates that the value should be rounded up. The only checking specifically mentioned is a comparison with an existing 8D table. These tables, so far as is known to the reviewer, are the only published ones to 10D with argument interval $0^g.01$. (Peters has such a table in manuscript form.) Linear interpolation gives 8D, but full accuracy is attainable only with quadratic interpolation.

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65[H, X].—JEAN PELTIER, *Resolution Numerique des Equations Algebriques*, Gauthier-Villars, Paris, 1957, iv + 244 p., 21 cm. Price $7.29.

The equations considered are polynomial equations of high degree in a single variable, and the treatment is from the standpoint of the desk-machine user. This means that to some extent the book is behind the times because it is usually quicker and cheaper to mail such problems to an organization equipped with a high-speed automatic computer and set of subroutines for iterative root-finding.

Only the most powerful methods are described and the reviewer is in agreement with the choice made. The opening chapters deal with elementary algebraic properties of the roots and the arithmetic operations required in the evaluation, multiplication and division of polynomials. Chapters 3 and 4 are devoted to the calculation of the moduli of the roots by Graeffe's root-squaring process and the subsequent determination of the phases by the "highest common factor" method. An omission here is the use of the latter method merely to resolve the finite number of ambiguities in the determination of the accurate phases from the root-squaring computations. Chapter 5 deals with iterative methods, and later chapters cover error propagaton and equations with complex coefficients. The recommended general computational procedure is summarized in the final chapter.

Many numerical examples are included, but little indication is given of checking procedures. In particular, the use of current checks on the root-squaring computations, which is vital in hand computing, does not appear to be mentioned.

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66[I].—L. N. KARMAZINA & L. V. KUROCHKINA, *Tablitsy interpolûtsionnykh koéffitsientov* (*Tables of interpolation coefficients*), Press of the Academy of Sciences of the USSR, Moscow, 1956, 376 p., 26.5 cm. Price 37.65 rubles.

These tables have been prepared at the Computing Centre of the Academy of Sciences. The coefficients are given throughout to 10 decimals. The fractional part of the interval being p, the following tables of Lagrangean coefficients are given: Table I, 3-point, $p = -1(0.001)1$; II, 4-point, $p = -1(0.001)2$; III, 5-point, $p = -2(0.001)2$; IV, 6-point, $p = -2(0.01)0(0.001)1(0.01)3$; V, 7-point, $p = -3(0.01) -1(0.001)1(0.01)3$; VI, 8-point, $p = -3(0.01)0(0.001)1(0.01)4$. Table VII gives 3 to 11-point Lagrangean coefficients for sub-tabulation to tenths.

All the coefficients for interpolation by differences are at interval 0.001, except Bessel's which are at 0.0001; coefficients up to those of the seventh difference are given throughout and the error is said not to exceed 6×10^{-11}. The even Bessel coefficients are those of mean, not double mean, differences (i.e. they are $2B_2$, $2B_4$, $2B_6$ in the notation of *Interpolation and Allied Tables*, London, 1956). Table VIII gives the Newton-Gregory coefficients, IX gives Bessel's, X gives Stirling's ($-0.5 \leq p \leq 0.5$), XI gives Everett's (to E_6). None of the tables of coefficients is provided with differences.

All the coefficients have been newly computed except the Lagrangean ones which, apart from the portion of the eight-point table at interval 0.01, have been taken from the National Bureau of Standards Tables (1944). One may hazard a guess that the Russian equivalent of the National accounting machines has eight registers; it is hard to see what else should make one stop at B_7; the predilection for Bessel's formula, which agrees with this reviewer's personal preference, is noteworthy; more striking is the absence from the Introduction of any mention of throw-back. In view of Chebyshev's connection with it, this is strange indeed.

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67[I, X].—H. E. SALZER, "Formulas for calculating Fourier coefficients," *Jn. Math. and Physics*, v. 36, 1957, pp. 96–98.

If a function is defined in the range $(0, 2\pi)$ as the polynomial of degree not exceeding eight which takes nine given values f_n at the nine equally spaced arguments $x = n\pi/4$, where $n = 0(1)8$, then all the coefficients in the Fourier expansion of the function in the range $(0, 2\pi)$ are defined, and may be calculated from the given values by using equations of the form

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\cos nx dx = a_1(f_0 + f_8) + a_2(f_1 + f_7) + a_3(f_2 + f_6) + a_4(f_3 + f_5) + a_5 f_4,$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx dx = b_1(f_0 - f_8) + b_2(f_1 - f_7) + b_3(f_2 - f_6) + b_4(f_3 - f_5).$$

The table on page 97 lists values of the nine coefficients a_i and b_i to 5D for $n = 0(1)24$, so that means are provided for calculating all Fourier coefficients up

to those of cos $24x$ and sin $24x$ inclusive. The values of a_i and b_i tabulated are said to be "most probably correct to within around $1\frac{1}{2}$ units in the 5th decimal, even though they cannot be absolutely guaranteed to less than $3\frac{1}{2}$ units in the 5th decimal."

A. F.

68[I].—H. E. Salzer & Peggy T. Roberson, *Table of Coefficients for Obtaining the Second Derivative without Differences*, Convari-Astronautics, San Diego 12, California, 1957, ix + 25 p., 26 cm., deposited in the *UMT* files.

If n function values are given at n equidistant arguments at interval h, this table lists coefficients for determining the value of the second derivative at any point in the range of width $(n - 1)h$. The tables relate to the cases $n = 5(1)9$, for which the Lagrange interpolation polynomials are of degrees 4(1)8 respectively, and their second derivatives consequently of degrees 2(1)6 respectively. For $n = 5(1)7$, the coefficients are given for hundredths of h; for $n = 8, 9$, they are given for tenths of h. The coefficients are in each case multiplied by an integer (6, 12, 360, 360, 10080 for $n = 5(1)9$ respectively) such that the tabulated values, instead of involving recurring decimals, are exact in a convenient number of decimals (4, 6, 8, 5, 6 respectively.) This useful table was calculated on the IBM 650 Magnetic Drum Data-Processing Machine.

A. F.

69[I].—H. O. Rosay, *Interpolation Coefficients* $\left(\frac{S}{K}\right)$ *for Newton's Binomial Interpolation Formula*, 20 tabulated sheets, 28 × 38 cm., deposited in the UMT File.

Binomial coefficients $\left(\frac{S}{K}\right)$, $K = 1(1)5$, $S = 0(.01)5$, 11D; however in many cases the last three digits are not reliable and accuracy is probably 8D.

The tables were computed on SWAC for use with the Gregory-Newton interpolation formula ([1], p. 96, equation 4.3.11). Similar tables for $K = 1(1)6$, $S = 0(.01)100$, 7D are in [2] according to [3].

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1. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.
2. G. Vega & J. A. Hülsse, *Sammlung Mathematischer Tafeln*, Weidmann, Leipzig, 1933.
3. A. V. Lebedev & R. M. Fedorova, *Spravochnik po Matematicheskim Tablitsam*, Moscow, 1956. [*MTAC*, Rev 49, v. 11, 1957, p. 104–106.]

70[L].—A. A. Abramov, *Tablitsy ln* $\Gamma(z)$ *v Kompleksnoĭ oblasti* (*Tables of ln* $\Gamma(z)$ *in a complex region*), Akad. Nauk SSSR, Moscow, 1953, 333 p., 26 cm. Price $3.50.

This volume gives 6D tables of the real and imaginary parts of $\ln \Gamma(x + iy)$ with second differences for $x = 1(.01)2$ and $y = 0(.01)4$. Real and imaginary parts are tabulated on opposite pages, each page having six columns corresponding to six values of x (the columns $x = 1.05(.05)1.95$ are repeated occurring as last

columns on one page and first columns on the following page) and 51 lines corresponding to fixed y (with repetitions for $y = .5(.5)3.5$). The introduction contains formulas for computation, error estimates, references both to books and tables (not including, however, Sibagaki's tables mentioned below), and a nomogram for $\frac{1}{2}(\xi - \eta)(\xi + \eta - 1)\Delta_2$. The volume is well printed.

For other tables of the gamma function in the complex domain see RMT **234** (*MTAC*, v. 2, p. 19), RMT **855** (*MTAC*, v. 5, p. 25–26), and RMT **1143** (*MTAC*, v. 7, p. 246). A small number of obvious misprints in the present work are corrected in pencil in the reviewer's copy. Comparison of 1300 values given both in the present tables and those by Sibagaki (RMT **1143**) showed about 100 discrepancies, mostly of one unit of the last decimal.

A. Erdélyi

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71[L].—Carl-Erik Fröberg, *Complete Elliptic Integrals*, Lund University Department of Numerical Analysis, Table No. 2, G. W. K. Gleerup, Lund, Sweden, 1957, 82 p., 25 cm. Price 12 Kr.

Ten decimal tables of the complete elliptic integrals of the first and second kinds as functions of the modulus. The functions tabulated are $K(k)$, $E(k)$, $K'(k) = K(k')$, $E'(k) = E(k')$ and two auxiliary functions

$$S(k) = K'(k) - \ln\frac{4}{k}, \quad S'(k) = K(k) - \ln\frac{4}{k'}.$$

The intervals are $k = 0.000(0.001)\ 0.900,\ 0.9000(0.0001)\ 1.0000$. Second differences, modified where necessary, are given for the ranges in which Everett's second order interpolation formula will give full accuracy. Elsewhere differences are omitted and auxiliary formulae are given, for use at the extreme ends of the table. In point of fact these auxiliary formulae could have been dispensed with had the author observed that while his function $S(k)$ is not interpolable at the given interval for $k < 0.005$, the function

$$K'(k) - \frac{2K(k)}{\pi}\ln\frac{4}{k}$$

is interpolable without restriction. Since the function $S(k)$ fails just where it is most needed, it is difficult to see the point of introducing it throughout the range of k, or indeed at all.

The tables were calculated in two ways namely by Landen's transformation and by power series but no information is given of the comparative labor of the two methods nor of the agreement of the results.

The argument which presents itself naturally in calculations with elliptic functions and integrals is k^2 rather than k. Furthermore the use of k^2 enables the size of the table to be halved without loss.

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72[L].—CARL-ERIK FRÖBERG & HANS WILHELMSSON, "Table of the function $F(a, b) = \int_0^\infty J_1(x)(x^2 + b^2)^{-\frac{1}{2}}dx$," *Fysiografiska Sallskapets I Lund Förhandlingar*, Bd. 27, 1957, p. 201–215.

The function $F(a, b)$ defined in the title is tabulated to 6D (which for most of the given values means 6S) for $a = .1(.1)2(.2)10$ and $b = 0(.1)2(.2)10$.

A short introduction mentions some physical applications of the function $F(a, b)$ and lists some limiting and special cases. Five-point Lagrangian interpolation is said to give full accuracy except for Max $(a, b) \leq 1$; to facilitate interpolation in this region, an auxiliary function $f(a, b)$ defined by

$$F(a, b) = \tfrac{1}{2}[(a^2 + b^2)^{\frac{1}{2}} - b] - [f(a, b)]^2$$

is tabulated for $a = .1(.1)1$, $b = 0(.1)1$, 6D.

The tables were calculated on SMIL, the electronic computer of Lund University, by numerical quadrature. For $b \leq 2$ a power series expansion was used as a check. The error is said to be in general less than 1 unit in the last place.

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73[L].—J. F. NICHOLAS, *A Table of $\int_0^x exp\,(-1/u)du$ for Small Values of x.* 28 Mimeographed pages, 22 × 16.5 cm., deposited in the UMT File.

Let $I(x) = \int_0^x e^{-1/u}du$ and $f(x) = x^{-2}e^{1/x}I(x)$. The table lists $f(x)$ and $I(x)$, $x = .01(.0005).05075$, 6S. The values given for $f(x)$ are said to be accurate except for small errors in the sixth place, the values given for $I(x)$ may be accurate to no more than four places for some values of the argument.

The tables are intended for use in connection with chemical and other reactions which are thermally activated and governed by an equation of the form $p = g(p)e^{-Q/RT}$.

They are related to other tables listed in the bibliography as well as a table whose existence is mentioned in a comment by Hastings [1]. They are also related to Hasting's rational approximations for the function $-\mathrm{Ei}(-x)$ listed in [1]. This relation follows from the identity listed by the author, $I(x) = xe^{-1/x} + \mathrm{Ei}^{-1/x}$.

The tables are reproduced photographically from machine tabulations, with a 7 page introduction reproduced photographically from typescript. A few copies may be available directly from the author at The Division of Tribophysics, Commonwealth Scientific and Industrial Research Organization, University of Melborne, Australia.

There is a minor misprint on page 3 of the introduction, where the author should have written $I(x) = xe^{-1/x} - \int_0^x u^{-1}e^{-1/u}du$; he has a + sign between the two main terms.

C. B. T.

1. CECIL HASTINGS, JR., JEANNE T. HAYWARD, & JAMES P. WONG, JR., *Approximations for Digital Computers*, Princeton, 1955, p. 187–190. See *MTAC*, v. 9, 1955, Review No. 56, p. 121–113.

74[L, P].—ROBERT L. STERNBERG, J. S. SHIPMAN, & S. R. ZOHN, *Table of the Bennett Functions $A_m(h)$ and $A_{mn}(h, k)$*, Laboratory for Electronics, Inc., Boston 14, Massachusetts, 18 p., 28 cm., deposited in *UMT* files.

The announced purpose of these undated and unpublished tables is to provide for the simple or multiple Fourier series expansion of the output from a cut-off power law rectifier when responding to a simple or multiple frequency oscillatory input.

The definitions of the functions are:

$$A_m(h) = \frac{2}{\pi}\int_{R_1} (\cos u - h) \cos mu \, du$$

$$\text{where } R_1\text{:} \quad \cos u \geq h, \quad 0 \leq u \leq \pi$$

$$A_{mn}(h, k) = \frac{2}{\pi^2}\int\int_{R_2} (\cos u + k \cos v - h) \cos mu \cos nv \, du \, dv,$$

$$\text{where } R_2\text{:} \quad \cos u + k \cos v \geq h, \quad 0 \leq u, \quad v \leq \pi.$$

Reference is made to two papers by W. R .Bennett [1].

$A_m(h)$ is given to 6D for $m = 0(1)9$ and $h = 0(.05)1$. $A_{mn}(h, k)$ is given to 6D for m, n non-negative integers on the range $0 = m + n = 4$ and $h = 0(0.1)2$, $k = 0.1(0.1)1$.

Computation was done on the IBM C.P.C. by from two to five different methods. All results are stated to have an absolute error of about 10^{-6} at most in absolute value.

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1. W. R. BENNETT, "New results in the calculation of modulation products," *Bell System Technical Journal*, v. 12, 1933, pp. 228–243; also "The biased ideal rectifier," *Bell System Technical Journal*, v. 26, 1947, pp. 139–169.

75[L, P].—CHIAO-MIN CHU, GEORGE C. CLARK, & STUART W. CHURCHILL, *Tables of Angular Distribution Coefficients for Light-Scattering by Spheres*, University of Michigan Press, 1957, xv + 58 p., 22 × 28 cm. Price $3.00.

Expressions for the differential scattering of light by a spherical particle are well known. Quite recently the present authors, following a suggestion of W. Hartel, derived forms which facilitate both numerical evaluation and application. The normalized differential scattering cross section is expressed by an expansion in Legendre polynomials. The coefficients are the quantities of interest here. They depend on two parameters, one the index of refraction of the medium surrounding the particle, the other a quantity, α, expressing the ratio of particle diameter to wave length.

The coefficients are evaluated for a series of fifteen refractive indices, ranging from 0.90 to 2.00 and also for the limit at infinity. The values of α are selected integral ones from the range 1–30. The accuracy is usually ±1 in the fifth decimal digit. The output of the computer is directly printed without transcription. It is

not explicitly stated to what extent the computations have been checked; such checks should be straightforward here. Perhaps Part II should have been labelled "Total Scattering Coefficients" rather than "Total Scattering Cross Sections." It would also have been interesting to include some of the computing times.

Together with the table of Legendre polynomials prepared by two of the authors (Clark and Churchill), calculations of differential scattering of light should now be simple and straightforward.

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76[M, P].—Gustav Doetsch, *Anleitung zum praktischen Gebrauch der Laplace-Transformation*, R. Oldenbourg Verlag, Munich 8, Germany, 1956, 198 p., 23 cm. Price DM 22.00.

The author of this work has written several well-known books on the Laplace transformation and its applications. The present book was undertaken at the request of the publishers and aims at providing a handbook of the Laplace transformation for the use of engineers, special consideration being given to the needs of the theory of servomechanisms. The presentation is lucid, and the style is free of mathematical sophistication; at the same time the statements are precise and mathematically sound. Proofs are not given, except where they contribute to a full understanding of the results. Conditions of validity are not only stated but also fully discussed; and the reader is often warned against neglecting to observe proper caution in using these methods.

After introducing the Laplace transformation in chapter 1, the so-called *rules* are discussed in chapter 2, applications to ordinary linear differential equations with constant coefficients, and to systems of such equations, are given in chapter 3, difference equations and recurrence relations are presented in chapter 4, partial differential equations in chapter 5, and integral equations and integral relations in chapter 6. The problem of inversion is discussed in chapter 7, and asymptotic expansions and stability in chapter 8.

An Appendix of some sixty pages, almost one third of the book, was compiled by Rudolf Herschel and gives transform pairs with emphasis on those needed in servoengineering and other engineering problems. There are 41 general formulas followed by nearly 300 transform pairs arranged according to image functions (116 rational functions, 109 algebraic and elementary transcendental functions, and 56 functions arising in the solution of differential equations).

Since the theoretical part covers a wide field in something like 130 pages, it is inevitable that not all topics are covered thoroughly. The presentation of ordinary differential equations with constant coefficients is very satisfactory as are, for a book of this kind, the unusually thorough discussions of the inversion problem and of asymptotic expansions. By comparison, the chapter on partial differential equations is somewhat meagre, and that on integral equations, skimpy. Differential and integral equations with time-lag are not discussed at all.

In the field which it does cover thoroughly, the book will prove a reliable guide

to the engineer whose mathematical training enables him to understand fully simple and precise mathematical statements, and who needs precise and unambiguous instructions for using Laplace transforms.

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77[S].—MANFRED VON ARDENNE, *Tabellen der Elektronenphysik Ionenphysik und Ubermikroscopie*, I. Band. Hauptgebiete, Deutscher Verlag der Wissenschaften, Berlin, 1956, xvi + 614 p., 27 cm. Price DM 74.

These two volumes contain a very complete collection of formulas, tables, references, and selections from the very extensive physical literature on electron physics, ion physics and the application of these subjects to microscopy. The material contained in these books relating to mathematics and mathematical methods is very meager. The latter topic is confined to a brief discussion (2 pages) of the Laplace equation and a reference to the Liebmann method for integrating this equation numerically.

Volume II contains six pages devoted to mathematical formulas and references. Of these, two pages are devoted to references, one page to a four place tables of exponential functions, two pages to formulas pertaining to statistics, one page to a figure illustrating the logarithmic scale used in the book, and one page giving the first few terms of the MacLaurin expansion of various simple functions.

A. H. T.

78[V, X].—B. ETKIN, *Numerical Integration Methods or Supersonic Wings in Steady and Oscillatory Motion*, UTIA Report No. 36, Institute of Aerophysics, University of Toronto, 1955, v + 37 p. + 22 figures + 4 tables, 28 cm. Available only on an exchange basis.

The paper deals with the problem of the determination of a perturbation velocity potential on supersonic wings with such a geometry that analytic methods are not applicable and, therefore, a numerical integration is unavoidable. It is based on the linearized theory of supersonic flow, and the cases of a supersonic as well as of a subsonic leading edge are considered.

The first part contains the results of the author's two previous papers on the problem of steady flow. The essential novelty consists of an extension of the method for oscillating wings. Insofar as the reduced frequency is concerned, three different stages of complexity appear. For a very slow oscillation the computation formulae are nearly identical with those for steady motion. If the frequency is sufficiently small to permit the linearization of the equations with respect to the frequency, then the formulae generated are still relatively simple. Finally, if this simplification is not possible, then the method still gives a system of equations, but the process for their numerical solution seems to be rather complicated.

Emphasis is placed on the presentation of the method in a form suitable for automatic digital computation. In some simple cases, such as those involving a

flat or a cambered delta wing, the numerical results are compared with the exact (linear) solutions and the agreement is found to be very good.

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79[W, Z].—RICHARD G. CANNING, *Electronic Data Processing for Business and Industry*, John Wiley and Sons Inc., New York, 1956, xi + 332 p., 23 cm. Price $7.00

The objective of this book "is the outline of a program of study and planning leading to the preparation of a proposal to top management." As the germ of his method, the author advocates (sic) systems engineering which he interprets for management personnel. The book is readable, many examples are given and the individual parts fit together quite well. Furthermore, the author recognized that hardware aspects could be expected to change (as they have) and accordingly qualified his material. This reviewer believes that the author has met his objective; this book probably will continue to prove useful to management personnel, particularly if it passes through revisions designed to keep it abreast of both technology and experience.

This reviewer does not wish to criticize this book on the basis of what it was not designed to provide. He does believe that it represents a reasonable set of case studies for clerical systems problems. The book is thus recommended for examination with the hope that such an observation process may stimulate or assist work on underlying numerical techniques or ingenious gadgetry, or both.

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80[W, Z].—RICHARD G. CANNING, *Installing Electronic Data Processing Systems*, John Wiley and Sons Inc., New York, 1957, vii + 193 p., 22 cm. Price $6.00.

Most managers who are considering Electronic Data Processing Systems, EDPS, should find this book of considerable value and interest. It should be of particular interest to those managers who have decided to include an EDPS in their organizations. The book is written in nontechnical language and assumes that the reader (the MANAGER) is relatively unacquainted with electronic computers. However, familiarity with such basic concepts as files, unit records, fields and some knowledge of programming is presumed.

The author contends throughout the book . . . "that top management should not remain so aloof from the *details* of electronic data processing. . . . It is time for management to recognize that EDP is a *management tool* and it is the responsibility of management to learn the advantages and limitations of this new tool. Perhaps the most effective way to learn about these new machines is to learn how to program a machine. Or to put it slightly differently, it is questionable whether a person really understands electronic data processing until he knows how to program." This reviewer concurs with the above statements.

This book consists essentially of a narrative description of the typical problems

encountered (and how they are solved) by an organization which is planning the installation of an EDPS. The organization is hypothetical (yet realistic); the problems encountered are not hypothetical, they are real problems. The discussion begins at a point in time where management has concluded a feasibility and application study and is beginning to plan for the installation of an EDPS. The discussion extends through a point in time (some three years later) where the installed system is in operation and the conversion period is almost complete. (The conversion means the changing over from the old operational system to the new electronic system.)

The EDPS organization (be it a section or branch or department) is first fitted in perspective, i.e., relative to the entire organizational structure. Management then begins plans for acquiring and training the personnel needed to staff the new EDPS section. Included are detailed job descriptions, qualifications and basis for selection for each of the various types of personnel required to staff the EDPS section. Approximately one half of the book discusses the personnel problems, i.e., acquisition, training, duties and performance. The case history reveals that the programming effort required should not be underestimated, viz., one might expect programming efforts to expend one man hour per one single address instruction coded and checked. The last half of the book discusses the problems encountered in the physical installation of the EDPS. This includes both hardware and personnel facilities and terminates with a thorough discussion of the problems encountered in the conversion period.

In summary, this book should be of considerable value to anyone involved in the planning, installation, or operation of an Electronic Data Processing System.

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81[W, Z].—G. Kozmetsky & P. Kircher, *Electronic Computers and Management Control*, McGraw-Hill, New York, 1956, 296 p., 23 cm. Price $5.00.

While it is difficult for this reviewer to judge, it would seem that this book, written primarily for the business executive and presupposing no technical training, succeeds in its purpose "to explain certain new developments which may have a greater influence on the management of enterprise than any other single group of events have had since the first industrial revolution."

A principal shortcoming is a complete lack of documentation. There are four footnotes and at least a dozen quotations from various pronouncements and articles; however, this reviewer did not see a single precise bibliographical reference within the text. In his opinion this greatly detracts from what seems to be a large collection of accurate and suggestive accounts of management experiences with computing equipments. So far as this reviewer could tell, the authors are reasonably meticulous in their limiting use of the present tense to computers and processes in being.

The authors' exposition on management control is capable and generally stimulating. Among the aspects described and related to computer applications

are the following: model-making, programming (planning), scheduling and feedback, and simulation.

In conclusion, this reviewer believes that the book would be of interest to those readers of this journal who wish to obtain some of the flavor of "business" data processing problems. He differs with the authors (p. 12) by believing that mathematicians must lead the way toward eventual resolution of difficulties in this particular area of numerical analysis.

W. H. Marlow

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82[W, X, Z].—Purdue University Computer Research Program, *Proceedings of Symposium I*, held at Purdue University, November 8 and 9, 1956, Purdue Research Foundation, iv + 76 p., 28 cm.

The book is a transcript of seven papers presented at Purdue University Computer Research Program, a symposium held November 8 and 9, 1956; the table of contents follows:

The Role of a University in an Industrial Society, by C. F. Kossack.
Administration of Research, by R. A. Morgen.
Reports on the Purdue Computer Research Program, by P. Brock.
The Purdue Compiler, by Sylvia Orgel.
Some Modern Linear Techniques in Practical Problems, by P. S. Dwyer.
A Re-Evaluation of Computing Equipment Needs, by S. N. Alexander.
Information Retrieval, by J. W. Mauchly.

C. B. T.

83[X].—William E. Milne, *Numerical Solution of Differential Equations*, John Wiley and Sons, Inc., New York, 1953, xi + 275 p., 23 cm. Price $7.25.

The work under review has in the years since its appearance been widely used both as a text and as a reference book. The book owes this success to several very attractive features. Eminently teachable and lavishly illustrated with worked examples, it leads the student along a gentle path. While providing a great deal of useful information, it seldom exposes the reader to the stern discipline of analytical rigor. Moreover, Professor Milne does not jump to the sterile conclusion that, because there is some good and some bad in every method, there is little to choose between them; on the contrary, he comes out with very definite opinions and thus offers firm guidance to the inexperienced worker in the field.

If in the following we shall comment critically on some aspects of Milne's book, we do not mean to belittle the author's achievement, which was quite substantial at the time the book was written. Rather we believe that our remarks will illustrate the enormous development which has taken place in the field of numerical computation during the last decade and which Professor Milne was one of the first to help to bring about.

Although high-speed machinery is mentioned at a few scattered places, it is probably fair to say that the whole outlook of Milne's book is dominated by the idea that all computations are carried out on a desk machine. The faithful disciple

of the book is imagined to work doggedly with pencil and paper, erasing predicted values when replacing them by corrected ones. This basic philosophy of the book is, in this reviewer's opinion, a serious detriment to its value as a guide for use in connection with automatic computers. The following remarks may serve to substantiate this statement.

The vexing problems that arise in the numerical integration of ordinary differential equations over a long range are not given adequate attention. The discussion of round-off is relegated to a rather superficial one-half page appendix. Although much attention is paid to the local error, no attempt is made to obtain realistic appraisals for the inherited error. Few numerical examples are carried further than ten steps.

In the sections on ordinary differential equations, which together comprise about 100 pages, the emphasis is on difference methods. Methods of Runge-Kutta type are discussed on two pages (without proofs), and everything is done to discourage their use. The simple second order Runge-Kutta methods are not even mentioned. While it is true that difference methods are somewhat less involved from the point of view of pencil and paper work, this advantage is hardly relevant from the point of view of computing machinery. Here all genuine one-step methods (such as the various Runge-Kutta methods or their modern modifications) enjoy the tremendous advantage of being self-starting (and thus cutting the length of any code in half) and of rendering trivial the operation of changing the basic interval.

Those who, in view of earlier publications of the author, expect the book to contain a wealth of relations involving finite differences will not be disappointed. Since for a given number of ordinates central difference formulas are the most accurate ones, these are particularly emphasized. Unfortunately these formulas are frequently unstable (in the sense of H. Rutishauser) if used for the integration of ordinary differential equations. A simple example may illustrate this. Milne discusses the classical Euler method ($y_{n+1} = y_n + hy'_n$) in two lines. He then proceeds immediately to a discussion of the midpoint rule ($y_{n+1} = y_{n-1} + 2hy'_n$) to which he devotes five pages. It so happens that the midpoint rule, quite apart from requiring a starting procedure, is unstable; it will not produce useful results for such a simple differential equation as $y' = -y$ over any reasonably long interval. Euler's method, on the other hand, will produce good, if not very accurate results for any differential equation. The celebrated Milne method (known in European countries as method of central differences, or simply as Simpson's rule) is also unstable, as was shown by Rutishauser [1]. While it is true that unstable methods usually produce good results over short ranges, their blind use in problems involving many steps can lead to disastrous effects.

The chapters on the solution of partial differential equations are handicapped by the fact that it was apparently not possible to give proper attention to several fundamental papers which became available shortly before the book was published. There is a long section on linear equations and matrices, intended to provide the student with the tools for solving implicit partial difference equations in particular those arising from elliptic problems. But the text does not mention the merits of the successive overrelaxation method, discussed independently by S. P. Frankel [2] and D. M. Young [3, 4], which has proved to be so eminently

successful in problems of this type. Again a great variety of partial difference operators is listed, but the discussion of numerical stability adds nothing to the 1928 Courant-Friedrichs-Lewy paper.

It has been pointed out that the general mathematical level of the book is elementary. To be specific, the level is well below the comparable book by L. Collatz [5]. While this is a desirable feature from the point of view of making the field accessible to students with a limited interest in mathematics, this reviewer has experienced a less desirable consequence of the lack of intellectually challenging material. To the bright mathematics student, if this is his first acquaintance with numerical analysis, the subject is made to appear a dull collection of recipes rather than a logical piece of mathematics. He has no opportunity to see the many interesting problems which merit his interest, and he will not feel attracted to a science which today more than ever needs his talents.

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1. VON HEINZ RUTISHAUSER, "Uber die instabilitat von mathoden zur integration gemohnlicher differentialgleichungen," *Zeit. f. ang. Math. u. Phys.*, v. 3, 1952, p. 65–74.

2. STANLEY P. FRANKEL, "Convergence rates of iterative treatments of partial differential equations," *MTAC*, v. 4, 1950, p. 65–75.

3. D. M. YOUNG, *Iterative Methods for Solving Partial Difference Equations of Elliptic Type*, Doctoral thesis, Harvard University, 1950.

4. D. M. YOUNG, "Iterative methods for solving partial difference equations of elliptic type," *Amer. Math. Soc., Trans.*, v. 76, 1954, p. 92–111.

5. L. COLLATZ, *Numerische Behandlung von Differentialgleichungen*, 2nd Ed., Springer-Verlag, Berlin, 1955.

84[Z].—JOHN ROBERT STOCK, *Die mathematischen Grundlagen für die Organisation der elektronischen Rechenmaschine der Eidgenössischen Technischen Hochschule*, Inst. f. angew. Math., *Mitt.*, Zurich, No. 6, 1956, 73 p., 23 cm. Price Sw. Fr. 7.30.

This seventy-three page book is the sixth in the current series on computers and applied mathematics prepared by the Institute for Applied Mathematics at the Eidgenossishe Technische Hochschule in Zurich, Switzerland.

The publication consists of a complete description of the ERMETH (Elektronische Rechenmaschine der Eidgenossishen Technischen Hochschule). In four Chapters, the mathematical and engineering characteristics, and the internal organization of the machine, are described. The first chapter presents a description of the mathematical characteristics, indicating number representation, instruction codes and machine operation. The second chapter gives the fundamentals of floating point, normalization and rounding, fixed point and index registers. The third chapter discusses the engineering fundamentals, including basic decision elements and arithmetic principles. The fourth chapter includes a discussion of internal organization and the manner in which the arithmetic unit performs addition, subtraction, multiplication and division. The control unit and error control methods are also described.

A large volume of information is given in the five appendices. Appendix I contains a complete description of the external features:

TECHNICAL DATA:

Drum storage capacity	10,000 words
Word length	16 decimal digits
Drum speed	6,000 RPM
Pulse repetition rate in storage	352 Kc
Pulse repetition rate in control and arithmetic unit	32 Kc
Cycle time	0.5 milliseconds
Average drum access time	5.0 milliseconds
Average access time in the index—instruction counter—register	2.5 milliseconds

OPERATION TIMES:

Add, excluding storage access, average	4 milliseconds
Multiply, excluding storage access, average	18 milliseconds
Divide, excluding storage access, average	28 milliseconds
Logical instruction	1 millisecond

COMPONENTS:

Tubes	1,900
Germanium diodes	7,000

Of these, 40% are in the arithmetic unit
25% are in the control unit
20% are in the storage switching unit
15% are in the input-output

WORD STRUCTURE:

Floating point number words		
Check digit (left end)		1
Exponent		3
Absolute value of mantissa (<10)		11
Sign (right end)		1
Fixed point number words		
Check digit (left end)		1
Absolute value of mantissa (<1)		14
Sign (right end)		
Instruction word		
Check digit (left end)		1
Left or first instruction		7
Operation	2	
Index	1	
Address	4	

Right or second instruction		7
Operation	2	
Index	1	
Address	4	
Sigh (right end)		1

Appendix II contains block diagrams, electrical schematics and logical diagrams, of the arithmetic unit, including a description of the micro-steps performed by the arithmetic unit during the execution of various types of instructions.

Appendix III gives similar information concerning the control unit. Included in Appendix IV is a detailed description of the networks contained in the adder.

Appendix V contains four different typical programs that have been prepared for the computer. A bibliography is included, however all of the references are dated 1953 or earlier.

The publication is written in clear, concise, easily understandable language. Although the ERMETH is a relatively slow computer (no high-speed storage other than the three arithmetic registers), the discussion and analysis of general computer organization and principles of operation are very interestingly presented. The reader will be quite pleased with the manner in which such subjects as number systems, arithmetic schemes, instruction codes, and programming techniques are handled. Sufficient information is given to permit the reader to code problems for the ERMETH. The book is worthwhile reading for anyone who wishes to familiarize himself with the general techniques and principles of computer construction, organization, and operation, even though the ERMETH itself may not be the reader's particular concern.

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85[Z].—TAKASHI KOJIMA, *The Japanese Abacus—Its Use and Theory*, Charles E. Tuttle Co., Tokyo, Japan, 1957, 102 p., 18 cm. Price $1.25.

There has been some interest in the abacus as the so-to-speak most elementary digital calculating instrument in use, but there has been no book in English which describes the abacus adequately.

This book appears to be the first of that kind.

Beginning with the story of the contest held in Japan in 1946 between operators of the Japanese abacus and the electro-mechanical desk calculator, it deals briefly with the history of the abacus in general, then goes on to explain step-by-step manipulations of beads of the Japanese abacus to do four arithmetic operations.

The book shows that by following various rules, the arithmetic operations (especially addition and subtraction) can be reduced to quite mechanical operations of the abacus requiring very small amounts of mental effort (it is only necessary to remember 10's complements of $1 - 9$ and 5's complements of $1 - 4$).

The book also contains some exercises for those who wish to acquire skill in manipulating the Japanese abacus.

The abacus is used quite extensively in Japanese business establishments. In the hands of experts, it is faster than the usual desk calculator for addition or subtraction and about as fast for multiplication or division of 10 to 12 digit numbers. The fact that the price of the Japanese abacus is approximately $3.00 may be one of the big reasons for its popularity among the small business establishments in Japan. However it must be borne in mind that to attain reasonable proficiency in the use of the abacus requires a number of months of practice.

The monograph provides interesting reading for those who are curious about the Japanese abacus.

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TABLE ERRATA

261.—LE CENTRE NATIONAL D'ÉTUDES DES TÉLÉCOMMUNICATIONS, *Tables des fonctions de Legendre associées*, Paris, 1952. [*MTAC*, v. 7, RMT **1110**, p. 178; *MTAC*, v. 8, Table Erratum **233**, p. 28.]

The authors report the following errors:

p. 38, ${P^0}_{9,6}(\cos 17^0) = -0.144118$ *and not* -0.144072
p. 42, ${P^0}_{9,6}(\cos 17^0) = -0.261274$ *and not* -0.261234
p. 82, ${P^1}_{8,6}(\cos 17^0) = -4.053574$ *and not* -4.053695
p. 86, ${P^1}_{9,6}(\cos 17^0) = -3.47195$ *and not* -3.47219
p. 136, ${P^2}_{9,6}(\cos 17^0) = 49.29970$ *and not* $-49,30035$.

They further report that there are a number of instances of poor printing, which might lead to confusion. The list follows:

p. 17, $P_3^0(\cos 80^0) = -0.2473\mathit{8}19$—the 8 is illegible
p. 18, ${P^0}_{3,9}(\cos 14^0) = 0.73381\mathit{9}5$—the 9 is poorly printed and could be mistaken for an 8
p. 60, ${P^1}_{3,1}(\cos 13^0) = -1.332\mathit{8}662$—the 8 is poorly printed
p. 78, ${P^1}_{7,9}(\cos 31^0) = 2.0\mathit{6}8962$—the 6 is poorly printed and could be mistaken for an 8.

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262.—CARL-ERIK FRÖBERG, *Hexadecimal Conversion Tables*, C. W. K. Gleerup, Lund, Sweden, 1957. [*MTAC*, Review **82**, v. 11, 1957, p. 208; *MTAC*, Table Erratum, v. 11, 1957, p. 309.]

On p. 10, *for* 0.38 30A3D 70A3D *4*0A3D *read* 0.38 30A3D 70A3D *7*0A3D.

B. ASKER

263.—D. R. Kaprekar, *Cycles of Recurring Decimals*, v. I. (From $N = 3$ to 161 and some other numbers.) Khare Wada, Deolali, India, 1950. Published by the author. [*MTAC*, Review **1126**, v. 7, 1953, p. 238.]

A rather comprehensive examination of this table from $N = 3$–163, inclusive revealed a number of errors not listed in the author's errata sheet at the back of the volume. Details of the errors noted by this investigator are given below.

In addition, there seem to be errata in Kaprekar's "errata" list. The correction he gives for page 16, line 20 should read 38–59–55 instead of 38–59–15; also the correction given for page 46, line 15 should read 115–93 instead of 115–98. The corrections given for page 8, line 8; also page 20, line 22 seem unnecessary; these "corrections" had already been made in the volume seen. Kaprekar's "correction" for page 24, line 13 is garbled; it should read 1, 1, 1, 8, 8, 8.

Page of Volume	N	Cycle	Pair of Lines in Cycle	Upper Line of Pair: Reads	Upper Line of Pair: Should Read	Lower Line of Pair: Reads	Lower Line of Pair: Should Read
6	33	9	1st	23–2	23–32		
	37	5	1st			8, 6, 2	1, 6, 2
15	101	25	1st		Entire cycle wrong—	See Note	(a)
16	107	2	2nd			8, 9, 0	9, 9, 0
17	111	12	1st	18–69–24	64–85–73	1, 6, 2	5, 7, 6 (b)
18	117	3	1st		Entire cycle wrong—	See Note	(c)
20	123	6	1st			4, 8, 9	0, 8, 9
22	133	4	1st	48–15–17	68–15–17		
48	159	6	1st	141–119–154	151–79–154		
		8	1st	115–77–134	119–77–134		
49	163	2	1st	64–114–162	44–114–162		
			2nd	122–179–138	122–79–138		

Notes

(a) Cycle 25 improperly duplicates Cycle 22; it should read 45–46–56–55
4 4 5 5

(b) Entire cycle wrong

(c) Cycle 3 should read 31–76–58–112–67–85
2 6 4 9 5 7

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264.—D. R. Kaprekar, *Cycles of Recurring Decimals*, v. II. (From $N = 167$ to 213 and many other numbers.) Khare Wada, Deolali, India, 1953. Published by the author. [*MTAC*, Review **1205**, v. 8, 1954, p. 148.]

In contrast to v. I, the v. II examined did not have an errata list. A rather comprehensive study of the tables from $N = 167$–213, inclusive, developed the errors listed below.

Page of Volume	N	Cycle	Pair of Lines in Cycle	Upper Line of Pair: Reads	Upper Line of Pair: Should Read	Lower Line of Pair: Reads	Lower Line of Pair: Should Read
1	167	1#	3rd	180–13–130	18–13–130		
			5th			3, 8, 1	2, 8, 1
				84–5–Blank	84–5–50	5, 0, blank	5, 0, 2
3	171	7	2nd			5, 8, 8	9, 8, 8
4	177	1	2nd			3, 9, 3	2, 9, 3
6	181	1##	5th			7, 6, 4	4, 6, 4
8	191	2	2nd	88–189–171	38–189–171		
	193	1#	1st			9, 5, 1	0, 5, 1
			3rd	103–125–92	109–125–92		
11	201	3	3rd			5, 8, 0	8, 8, 0
12	203	1	3rd			5, 3, 3	4, 3, 3
		2	7th	143–39–187	146–39–187		
	207	1	2nd	131–118–145	136–118–145		
15	211	4	2nd			9, 6, 3	7, 6, 3

Notes
First half
Second half

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NOTES

National Bureau of Standards and National Science Foundation Training Program in Numerical Analysis for Senior University Staff Spring 1959

The National Bureau of Standards is planning, conditional upon support by the National Science Foundation, to hold its second Training Program in Numerical Analysis for Senior University Staff.

The purpose of this program is to give regular university staff a training in numerical analysis which will enable them to direct the operation of a university computing center, and to organize training and research in numerical analysis on their return to their own institutions. It will occupy the whole of the second semester of the academic year 1958–1959 (from February 9 to June 5) and has been arranged for that time so that participants may become familiar with the details of their own computing equipment during the following summer and be able to conduct courses in the academic year 1959–1960.

Applications must be received not later than October 15, 1958. They should be addressed to:

Dr. Philip J. Davis, Chief
Numerical Analysis Section
Applied Mathematics Division
National Bureau of Standards
Washington 25, D. C.

and should include:

1. The academic history of applicant and a list of his publications.
2. A statement about the computational program of the institution.

CORRIGENDA

A. J. M. Hitchcock, "Polynomial approximations to bessel functions of order zero and one and to related functions," *MTAC*, v. 11, 1957, p. 86, line 2 from bottom,

for $$\bar{J}_0(x) = 1 - \sqrt{\frac{\pi}{2x}}\left(\bar{P}_0(x)\cos\left(x + \frac{\pi}{4}\right) - \bar{Q}_0(x)\sin\left(x + \frac{\pi}{4}\right)\right)$$

read $$\bar{J}_0(x) = 1 - \sqrt{\frac{2}{\pi x}}\left(\bar{P}_0(x)\cos\left(x + \frac{\pi}{4}\right) - \bar{Q}_0(x)\sin\left(x + \frac{\pi}{4}\right)\right)$$

s of
86,